

# HOMOTOPY MOMENT MAPS

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**ABSTRACT.** Associated to any manifold equipped with a closed form of degree  $> 1$  is an  $L_\infty$ -algebra which acts as a higher/homotopy analog of the Poisson Lie algebra of functions on a symplectic manifold. In order to study Lie group actions on these manifolds, we introduce a theory of homotopy moment maps. Such a map is a  $L_\infty$ -morphism from the Lie algebra of the group into the higher Poisson Lie algebra which lifts the infinitesimal action. We establish a relationship between homotopy moment maps and equivariant de Rham cohomology, and analyze the obstruction theory for the existence of such maps. This allows us to easily and explicitly construct a large number of examples. These include results concerning group actions on loop spaces and moduli spaces of flat connections. Relationships are also established with previous work by others in classical field theory, algebroid theory, and dg geometry. Furthermore, we use our theory to geometrically construct various  $L_\infty$ -algebras as higher central extensions of Lie algebras, in analogy with Kostant's quantization theory. In particular, the so-called 'string Lie 2-algebra' arises this way.

## CONTENTS

1. Introduction	1
2. Preliminaries	5
3. $L_\infty$ -algebras	6
4. $L_\infty$ -algebras from closed differential forms	8
5. Homotopy moment maps	12
6. Equivariant de Rham cohomology	13
7. Closed 3-forms	18
8. Examples	21
9. Obstructions and central extensions	25
10. Moduli spaces of flat connections	32
11. Loop spaces	38
12. Relation to other works	40
13. Concluding remarks	43
Appendix A. Explicit formulas for $L_\infty$ -morphisms	44
References	52

## 1. INTRODUCTION

This paper represents part of a larger project which involves studying the symmetries of manifolds equipped with a closed differential form. The motivation for this work stems from the desire to have a more conceptual understanding of the role these manifolds play in differential cohomology, generalized geometry, and certain topological field theories. In our approach, we view such manifolds as generalizations of symplectic manifolds.

As a first step, we consider symmetries arising from a Lie group acting on a manifold by diffeomorphisms which preserve a closed differential form. A key component of our formalism is the 'homotopy moment map'. This is a natural generalization of the moment map used to study symmetries in symplectic geometry. However, unlike their symplectic counterparts, our moment maps

do not correspond to morphisms between Lie algebras. Instead, they are morphisms between objects called ‘ $L_\infty$ -algebras’, which can be thought of as homotopy-theoretic upgrades of Lie algebras. More precisely, an  $L_\infty$ -algebra is a graded vector space equipped with a skew-symmetric bracket which satisfies the Jacobi identity up to coherent homotopy. The coherent homotopy is given as part of the data by an infinite sequence of higher degree multi-linear brackets which satisfy additional Jacobi-like identities.  $L_\infty$ -algebras with underlying vector spaces concentrated in the first non-positive  $n$ -degrees are often called ‘Lie  $n$ -algebras’. In particular, a Lie 1-algebra is an ordinary Lie algebra.

Morphisms between  $L_\infty$ -algebras are not just linear maps which preserve the brackets. This definition is too strict. Rather, a morphism is an infinite collection of multi-linear maps which preserve the brackets up to, again, coherent homotopy. We emphasize that the morphisms between  $L_\infty$ -algebras are what matter to us, not so much the algebras themselves. This coincides with the point of view that  $L_\infty$ -algebras are examples of ‘higher structures’, i.e. objects of a higher category, in which the morphisms between two objects form something more akin to a topological space rather than a set.

Perhaps it seems strange that these higher structures should appear when studying something as classical as actions of Lie groups on manifolds. To understand why they are needed, we have to first recall some facts concerning symmetries in symplectic geometry.

**1.1. Symplectic geometry.** The important infinitesimal symmetries of a symplectic manifold correspond to the flows of the Hamiltonian vector fields. These form a Lie algebra whose bracket is the usual commutator of vector fields. The space of smooth functions is also a Lie algebra, whose bracket is specified by the symplectic 2-form. We will call this ‘the Poisson Lie algebra’. If the manifold is connected (we always assume this is the case), then Kostant [19] showed that the Poisson Lie algebra is characterized as a particular extension of the Lie algebra of Hamiltonian vector fields by  $\mathbb{R}$ . The 2-cocycle representing this central extension is determined by the symplectic form.

Now suppose we have a Lie group  $G$ , with Lie algebra  $\mathfrak{g}$ , acting on the manifold via diffeomorphisms which preserve the symplectic form. Assume further that the associated infinitesimal action is given by a Lie algebra morphism from  $\mathfrak{g}$  to the Hamiltonian vector fields. A ‘moment map’ for the action corresponds to a lift of this Lie algebra morphism to the central extension given by the Poisson Lie algebra<sup>1</sup>. Whether a moment map exists or not for a particular  $G$ -action is an important question in symplectic geometry. It can be thought of as the symplectic analog of determining when a projective representation of  $G$  lifts to a linear one.

The relationship between symmetries in symplectic geometry and representation theory is made explicit via ‘geometric quantization’. If the symplectic form represents an integral cohomology class, then it corresponds to the curvature of a principal  $U(1)$ -bundle equipped with a connection. In this case, the Poisson Lie algebra is isomorphic to a Lie algebra consisting of the  $U(1)$ -invariant vector fields on the bundle whose flows preserve the connection. This Lie algebra acts naturally as differential operators on sections of the associated Hermitian line bundle. Hence, if there is a  $G$ -action on the symplectic manifold, then a moment map for this action gives a representation of the Lie algebra  $\mathfrak{g}$  on the space of sections. In certain cases, this action integrates to a global  $G$ -action.

If no moment map exists, then Kostant’s construction produces non-trivial central extensions of both  $\mathfrak{g}$  and  $G$  which naturally act on the space of sections of the Hermitian line bundle. Many important Lie groups can be constructed this way e.g. central extensions of loop groups, as well as the Heisenberg and Bott-Virasoro groups [5, Sec. 2.4.].

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<sup>1</sup>Technically, this lift is not the moment map, but rather the “co-moment map”.

**1.2. “Higher” symplectic geometry.** Let us return to the more general case, and consider a manifold equipped with a closed form of degree  $n + 1 > 2$ . Such a manifold also has Hamiltonian vector fields, and these form a Lie algebra just as they do in symplectic geometry. To pursue the analogy further, one might try to construct a central extension of the Hamiltonian vector fields using the closed  $(n + 1)$ -form. Unlike the symplectic case, the form does not induce a skew-symmetric bracket on functions. But it does on a particular subspace of  $(n - 1)$ -forms, called Hamiltonian  $(n - 1)$ -forms. This bracket, however, fails to satisfy the Jacobi identity.

This lack of a genuine Lie bracket is the reason why  $L_\infty$ -algebras appear. In previous work, the second author considered such manifolds when the closed  $(n + 1)$ -form satisfied a mild non-degeneracy condition. These are called ‘ $n$ -plectic’ or ‘multisymplectic’ manifolds. He associated to such a manifold a Lie  $n$ -algebra whose underlying vector space consists of the Hamiltonian  $(n - 1)$ -forms and all other forms of lower degree. Its brackets are completely determined by the  $(n + 1)$ -form and the de Rham differential. Later, the third author showed that the non-degeneracy assumption is not necessary for the construction, and therefore any ‘pre- $n$ -plectic’ manifold has such a Lie  $n$ -algebra.

In this work, we slightly generalize these previous constructions and associate to any manifold equipped with a closed  $(n + 1)$ -form a ‘Poisson Lie  $n$ -algebra’. In analogy with Kostant’s central extension for a connected symplectic manifold, the Poisson Lie  $n$ -algebra of a pre  $n$ -plectic manifold can also be characterized uniquely, up to equivalence, assuming certain topological conditions are satisfied. More specifically, the Poisson Lie  $n$ -algebra is a “higher central extension” of the Lie algebra of Hamiltonian vector fields, and this extension is represented by a degree  $(n + 1)$ -cocycle determined by the closed form. The (pre-) symplectic case is recovered when  $n = 1$ , i.e. the Poisson Lie 1-algebra is the aforementioned Poisson Lie algebra.

If a Lie group  $G$  acts on a pre- $n$ -plectic manifold and the infinitesimal action induces a Lie algebra morphism between  $\mathfrak{g}$  and the Hamiltonian vector fields, then we define a ‘homotopy moment map’, or just ‘moment map’ for short, to be a lift of this Lie algebra morphism to an  $L_\infty$ -morphism from  $\mathfrak{g}$  to the Poisson Lie  $n$ -algebra. At first sight, this definition may seem too abstract or technical to be useful. However, thanks to some of the tools developed here, we can easily and systematically construct such maps and therefore produce a large variety of interesting examples.

A REMARK ON EXPOSITION: This paper is aimed at a broad audience of geometers and topologists. We assume the reader has essentially no expertise in homotopical algebra, and perhaps only a slight familiarity with higher geometric structures.

**1.3. Summary of results.** We begin with a quick introduction to  $L_\infty$ -algebras in Sec. 3 and leave the more technical aspects to the appendix. We review the necessary background on  $n$ -plectic geometry, Hamiltonian vector fields, and the Poisson Lie  $n$ -algebra in Sec. 4.

We introduce the homotopy moment map in Sec. 5. If  $G$  acts on a pre- $n$ -plectic manifold  $(M, \omega)$  such that  $\mathfrak{g}$  acts via Hamiltonian vector fields, then a moment map is a lift of this action to an  $L_\infty$ -morphism from  $\mathfrak{g}$  to the Poisson Lie  $n$ -algebra of  $(M, \omega)$ . The precise definition is given in Def. 5.1. In almost all examples, this morphism is not “strict”. For  $n = 1$ , we recover the usual notion of a (co)-momentum map in pre-symplectic geometry.

*Equivariant de Rham cohomology.* In Sec. 6, we present our first main result which implies that any degree 3 cocycle in equivariant de Rham cohomology gives a homotopy moment map. This generalizes the important relationship between moment maps in symplectic geometry and degree 2 cocycles in equivariant cohomology. We can also produce moment maps from higher degree cocycles of a particularly nice form. Many of our examples arise in this way. We have strong evidence that every equivariant cocycle corresponds to a homotopy moment map. However, a nice proof of this will require some additional machinery, so we postpone this generalization to a later publication [14].

*Closed 3-forms.* The first truly new (i.e. non-symplectic) examples will arise on manifolds equipped with a closed 3-form. Such manifolds also play an important role in generalized geometry and the theory of gerbes. So in Sec. 7, we focus on aspects specific to this case.

*Basic examples.* We then present some basic examples in Sec. 8. These include:

- Exact pre-plectic forms: This generalizes familiar results in symplectic geometry involving  $G$ -invariant symplectic potentials. Special cases include  $G$ -actions on exterior powers of cotangent bundles, and the action of  $\mathrm{SO}(n)$  on  $\mathbb{R}^n$  equipped with the usual volume form.
- $\mathrm{SO}(3)$ -action on the 3-sphere: This generalizes the Hamiltonian circle action on  $S^2$  whose moment map corresponds to the “height function” along the  $z$ -axis.
- Compact Lie groups: The Cartan 3-form on such a group is invariant under conjugation and can be uniquely extended to an equivariant closed 3-form. This gives a moment map for the adjoint action. We point out a relationship between this moment map and certain quasi-Hamiltonian  $G$ -spaces.

*Obstructions and higher central extensions.* In order to produce more examples, we study in Sec. 9 the obstructions to lifting a  $G$  action to a homotopy moment map. The results we present here are natural generalizations of the symplectic ones. The existence of a moment map for a  $G$ -action on a (connected) pre- $n$ -plectic manifold implies that a degree  $(n+1)$  class  $[c]$  in Lie algebra cohomology is trivial. Conversely, if  $[c] = 0$  and  $M$  satisfies certain topological assumptions, then we can always construct a moment map lifting the action (Thm. 9.7).

If  $[c] \neq 0$ , then in Sec. 9.3 we show how to construct a  $L_\infty$ -morphism not from  $\mathfrak{g}$ , but rather from a Lie  $n$ -algebra  $\widehat{\mathfrak{g}}$ . The Lie  $n$ -algebra  $\widehat{\mathfrak{g}}$  is built using a representative of  $[c]$  and plays the role of a (non-trivial) higher central extension of  $\mathfrak{g}$ . This suggests a new way to geometrically construct Lie  $n$ -algebras. For example, via this construction we recover the string Lie 2-algebra  $\mathbf{string}(\mathfrak{g})$ , which plays an interesting role in elliptic cohomology.

*Moduli spaces and loop spaces.* In Sec. 10, we use the results of Sec. 9 to produce a more sophisticated example of a homotopy moment map on an infinite-dimensional manifold. If  $P$  is a principal  $G$ -bundle on a  $(n+1)$ -dimensional compact oriented manifold, then a degree  $(n+1)$  invariant polynomial on  $\mathfrak{g}$  gives a pre- $n$ -plectic structure on the space of connections of  $P$ . We show that this  $(n+1)$ -form is invariant under the action of the gauge group, and that this action admits a moment map. If the  $(n+1)$ -form is actually  $n$ -plectic, then we can perform a Marsden-Weinstein reduction procedure to obtain the moduli space of flat connections, endowed with a pre- $n$ -plectic form. This generalizes the well-known Atiyah-Bott construction in symplectic geometry for  $G$ -bundles over Riemann surfaces.

We continue to focus on infinite-dimensional examples in Sec. 11. There we show that a homotopy moment map for a  $G$  action on a pre-2-plectic manifold  $(M, \omega)$  can be transgressed to an ordinary moment map on the pre-symplectic loop space  $(LM, L\omega)$ , where  $L\omega$  is transgression of  $\omega$ , and the action of  $G$  on  $LM$  is “point-wise”. This gives an example of how the higher geometry on  $M$  interacts with the classical geometry on  $LM$ .

*Comparisons with other work.* Numerous generalizations of moment maps already exist in the literature. In Sec. 12, we describe some relationships between homotopy moment maps and related work done by others. In particular, we consider: multi-momentum maps studied by a variety of authors in multisymplectic field theory [8, 16], the multi-moment maps of Madsen and Swann [21], Bursztyn, Cavalcanti, and Gualtieri’s work on group actions and Courant algebroids, and Uribe’s work on group actions and dg-manifolds [32].

Finally, we conclude in Sec. 13 with some open questions.

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## 2. PRELIMINARIES

Here we list the notation and conventions used throughout the paper. We also give a brief review of the Cartan calculus for multi-vector fields.

**2.1. Graded linear algebra.** Let  $V$  be a  $\mathbb{Z}$  graded vector space. For any  $k \in \mathbb{Z}$ ,  $V[k]$  is the graded vector space

$$V[k]^i = V^{i+k}.$$

Let  $x_1, \dots, x_n$  be elements of  $V$  and  $\sigma \in \mathcal{S}_n$  a permutation. The **Koszul sign**  $\epsilon(\sigma) = \epsilon(\sigma; x_1, \dots, x_n)$  is defined by the equality

$$x_1 \cdots x_n = \epsilon(\sigma; x_1, \dots, x_n) x_{\sigma(1)} \cdots x_{\sigma(n)},$$

which holds in the free graded commutative algebra generated by  $V$ , with product denoted by concatenation of elements. Given  $\sigma \in \mathcal{S}_n$ , let  $(-1)^\sigma$  denote the usual sign of a permutation. Note that  $\epsilon(\sigma)$  does not include the sign  $(-1)^\sigma$ .

We say  $\sigma \in \mathcal{S}_{p+q}$  is a **(p, q)-unshuffle** iff  $\sigma(i) < \sigma(i+1)$  whenever  $i \neq p$ . The set of  $(p, q)$ -unshuffles is denoted by  $\text{Sh}(p, q)$ . For example,  $\text{Sh}(2, 1)$  is the set of cycles  $\{(1), (23), (123)\}$ . If  $V$  and  $W$  are graded vector spaces, a linear map  $f: V^{\otimes n} \rightarrow W$  is **skew-symmetric** iff

$$f(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = (-1)^\sigma \epsilon(\sigma) f(v_1, \dots, v_n),$$

for all  $\sigma \in \mathcal{S}_n$ . The degree of an element  $x_1 \otimes \cdots \otimes x_n \in V^{\otimes \bullet}$  of the graded tensor algebra generated by  $V$  is defined to be  $|x_1 \otimes \cdots \otimes x_n| = \sum_{i=1}^n |x_i|$ .

Finally, the following sign occurs frequently, so we give its own notation. For an integer  $k$  define:

$$\varsigma(k) = -(-1)^{\frac{k(k+1)}{2}}.$$

So for  $k = 1, 2, 3, 4, 5, \dots$  we have  $\varsigma(k) = 1, 1, -1, -1, 1, \dots$ . Notice that  $\varsigma(k-1)\varsigma(k) = (-1)^k$  for all  $k$ .

**2.2. Group actions.** Throughout this paper  $G$  denotes a Lie group and  $\mathfrak{g}$  its Lie algebra. A  $G$  action on a manifold  $M$  is always from the left, unless stated otherwise. Our convention for the induced action on forms is the one given in [17, Sec. 2.1]. Namely,  $G$  acts on  $\Omega^\bullet(M)$  from the left via inverse pullback

$$g \cdot \omega \mapsto \phi_{g^{-1}}^* \omega,$$

where  $\phi_g$  is the diffeomorphism corresponding to  $g$ . We denote the corresponding infinitesimal action of the Lie algebra  $\mathfrak{g}$  by the map

$$(1) \quad v_- : \mathfrak{g} \rightarrow \mathfrak{X}(M), \quad x \mapsto v_x,$$

where

$$v_x|_p = \frac{d}{dt} \exp(-tx) \cdot p|_{t=0} \quad \forall p \in M.$$

We call  $v_-$  the **fundamental vector field** associated to the  $G$  action. Note that it is *minus* the infinitesimal generator associated to the  $G$  action, and hence it is a morphism of Lie algebras.

**2.3. Cartan calculus.** Let  $\mathfrak{X}(M)$  be the  $C^\infty(M)$ -module of vector fields on a manifold  $M$  and let

$$\mathfrak{X}^{\bullet}(M) = \bigoplus_{k=0}^{\dim M} \Lambda^k \mathfrak{X}(M)$$

be the graded commutative algebra of multivector fields. The **Schouten bracket**  $[\cdot, \cdot]: \mathfrak{X}^{\bullet}(M) \times \mathfrak{X}^{\bullet}(M) \rightarrow \mathfrak{X}^{\bullet}(M)$  is a degree  $-1$  Lie bracket which satisfies the graded Leibniz rule with respect to the wedge product. The Schouten bracket of two decomposable multivector fields  $u_1 \wedge \cdots \wedge u_m, v_1 \wedge \cdots \wedge v_n \in \mathfrak{X}^{\bullet}(M)$  is

$$(2) \quad [u_1 \wedge \cdots \wedge u_m, v_1 \wedge \cdots \wedge v_n] = \sum_{i=1}^m \sum_{j=1}^n (-1)^{i+j} [u_i, v_j] \wedge u_1 \wedge \cdots \wedge \hat{u}_i \wedge \cdots \wedge u_m \wedge v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_n,$$

where  $[u_i, v_j]$  is the usual Lie bracket of vector fields.

The **interior product** of a decomposable multivector field  $v_1 \wedge \cdots \wedge v_n$  with  $\alpha \in \Omega^{\bullet}(M)$  is

$$(3) \quad \iota(v_1 \wedge \cdots \wedge v_n) \alpha = \iota_{v_n} \cdots \iota_{v_1} \alpha,$$

where  $\iota_{v_i} \alpha$  is the usual interior product of vector fields and differential forms. The interior product of an arbitrary multivector field is obtained by extending by  $C^\infty(M)$ -linearity.

The **Lie derivative**  $\mathcal{L}_v$  of a differential form along a multivector field  $v \in \mathfrak{X}^{\bullet}(M)$  is the graded commutator of  $d$  and  $\iota(v)$ :

$$(4) \quad \mathcal{L}_v \alpha = d\iota(v) \alpha - (-1)^{|v|} \iota(v) d\alpha,$$

where  $\iota(v)$  is considered as a degree  $-|v|$  operator.

The last identity we will need is for the graded commutator of the Lie derivative and the interior product. Given  $u, v \in \mathfrak{X}^{\bullet}(M)$ , it follows from [13, Proposition A3] that

$$(5) \quad \iota([u, v]) \alpha = (-1)^{(|u|-1)|v|} \mathcal{L}_u \iota(v) \alpha - \iota(v) \mathcal{L}_u \alpha.$$

### 3. $L_\infty$ -ALGEBRAS

In this section we briefly review  $L_\infty$ -algebras and explicitly describe  $L_\infty$ -morphisms for the special cases considered in this paper.

**Definition 3.1** ([20]). An  **$L_\infty$ -algebra** is a graded vector space  $L$  equipped with a collection

$$\left\{ l_k: L^{\otimes k} \rightarrow L \mid 1 \leq k < \infty \right\}$$

of graded skew-symmetric linear maps with  $|l_k| = 2 - k$  such that the following identity holds for  $1 \leq m < \infty$ :

$$(6) \quad \sum_{\substack{i+j=m+1, \\ \sigma \in \text{Sh}(i, m-i)}} (-1)^\sigma \epsilon(\sigma) (-1)^{i(j-1)} l_j(l_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(m)}) = 0.$$

In the appendix (Sec. A.3), we recall how any  $L_\infty$ -algebra  $(L, l_k)$  corresponds to a certain kind of graded coalgebra  $C(L)$  equipped with a coderivation  $Q$  which satisfies the identity

$$Q \circ Q = 0.$$



This identity is the origin of Eq. (6). But it is easy to see that for small values of  $m$  that Eq. (6) is a “generalized Jacobi identity” for the multi-brackets  $\{l_k\}$ . For  $m = 1$ , it implies that the degree 1 linear map  $l_1$  satisfies

$$l_1 \circ l_1 = 0$$

and hence every  $L_\infty$ -algebra  $(L, l_k)$  has an underlying cochain complex  $(L, l_1)$ .

**Definition 3.2.** An  $L_\infty$ -algebra  $(L, \{l_k\})$  is a **Lie  $n$ -algebra** iff the underlying graded vector space  $L$  is concentrated in degrees  $0, -1, \dots, 1 - n$ .

Note that if  $(L, \{l_k\})$  is a Lie  $n$ -algebra, then by degree counting  $l_k = 0$  for  $k > n + 1$ . An ordinary Lie algebra is the same as a Lie 1-algebra.

**3.1.  $L_\infty$ -morphisms.** The following definition may at first seem satisfactory:

**Definition 3.3** ([20]). Let  $(L, l_k)$  and  $(L', l'_k)$  be  $L_\infty$ -algebras. A degree 0 linear map  $f: L \rightarrow L'$  is a **strict  $L_\infty$ -morphism** iff

$$(7) \quad l'_k \circ f^{\otimes k} = f \circ l_k \quad \forall k \geq 1.$$

However, this definition of  $L_\infty$ -morphism does not reflect the higher structure naturally residing within the theory. Indeed, the better definition [20, Remark 5.3] uses the aforementioned relationship between  $L_\infty$ -algebras and dg-coalgebras. This ultimately gives the collection of morphisms between two  $L_\infty$ -algebras the structure of a simplicial set [18], which then allows one to consider homotopies between morphisms, homotopies between homotopies, and so on. It is not necessary for the reader to understand these remarks precisely. However, we emphasize that the flexibility provided by this higher structure is what allows us to produce the many explicit examples of homotopy moment maps considered in this paper.

**Definition 3.4.** An  **$L_\infty$ -morphism** between  $L_\infty$ -algebras  $(L, l_k)$  and  $(L', l'_k)$  is a morphism  $F: (C(L), Q) \rightarrow (C(L'), Q')$  between their corresponding differential graded (dg) coalgebras. That is,  $F$  is a morphism between the graded coalgebras  $C(L)$  and  $C(L')$  such that

$$(8) \quad F \circ Q = Q' \circ F.$$

It turns out that an  $L_\infty$ -morphism between  $(L, l_k)$  and  $(L', l'_k)$  corresponds to an infinite collection of graded skew-symmetric ‘structure maps’

$$f_k: L^{\otimes k} \rightarrow L' \quad 1 \leq k < \infty,$$

where  $|f_k| = 1 - k$ , and such that a complicated compatibility relation with the multi-brackets is satisfied. In particular, the degree zero map  $f_1$  must be a morphism between the underlying complexes  $(L, l_1)$  and  $(L', l'_1)$ :

$$f_1 l_1 = l'_1 f_1.$$

The compatibility relation, in the language of coalgebras, corresponds exactly to Eq. (8). Strict morphisms in the sense of Def. 3.3 correspond to the special case when  $f_k = 0$  for  $k \geq 2$ . (See Prop. A.5 for more details.) Outside of Sec. A.4, we shall mildly abuse notation and denote a  $L_\infty$ -morphism via its structure maps as

$$(f_k): (L, l_k) \rightarrow (L', l'_k).$$

$L_\infty$ -morphisms are composable in the usual sense, and hence one can speak of the category of  $L_\infty$ -algebras without explicitly describing the higher structure mentioned above.

**Definition 3.5.** We denote by  $\text{Lie}_\infty$  the category whose objects are  $L_\infty$ -algebras (Def. 3.1) and whose morphisms are  $L_\infty$ -morphisms (Def. 3.4).

The following is the correct notion of equivalence between  $L_\infty$ -algebras which reflects the aforementioned homotopical structure between morphisms.

**Definition 3.6.** A morphism  $(f_k): (L, l_k) \rightarrow (L', l'_k)$  of  $L_\infty$ -algebras is a  **$L_\infty$ -quasi-isomorphism** iff the morphism of complexes

$$f_1: (L, l_1) \rightarrow (L', l'_1)$$

induces an isomorphism on the cohomology:

$$H^\bullet(f_1): H^\bullet(L) \xrightarrow{\cong} H^\bullet(L').$$

**Remark 3.7.**  $L_\infty$ -quasi-isomorphisms induce an equivalence relation on the category of  $L_\infty$ -algebras. Indeed, such a morphism between two  $L_\infty$ -algebras exists if and only if the algebras are homotopy equivalent (in the sense of Quillen model categories [10] [18]). Roughly speaking, the situation here is analogous to the Whitehead theorem for weak homotopy equivalences between CW complexes.

**3.2. Morphisms from Lie algebras to  $L_\infty$ -algebras.** In this paper, we will be particularly interested in  $L_\infty$ -morphisms from a Lie algebra to a Lie  $n$ -algebra  $(L', l'_k)$  with the following property:

$$(P) \quad \forall k \geq 2 \quad l'_k(x_1, \dots, x_k) = 0 \quad \text{whenever} \quad \sum_{i=1}^k |x_i| < 0.$$

The following characterization is proven in the appendix (Cor. A.7).

**Proposition 3.8.** *If  $(\mathfrak{g}, [\cdot, \cdot])$  is a Lie algebra and  $(L', l'_k)$  is a Lie  $n$ -algebra satisfying property (P), then the graded skew-symmetric maps*

$$f_k: \mathfrak{g}^{\otimes k} \rightarrow L', \quad |f_k| = 1 - k, \quad 1 \leq k \leq n$$

*are the components of an  $L_\infty$ -morphism  $\mathfrak{g} \rightarrow L'$  if and only if  $\forall x_i \in \mathfrak{g}$*

$$(9) \quad \sum_{1 \leq i < j \leq k} (-1)^{i+j+1} f_{k-1}([x_i, x_j], x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_k) \\ = l'_1 f_k(x_1, \dots, x_k) + l'_k(f_1(x_1), \dots, f_1(x_k)).$$

*for  $2 \leq k \leq n$  and*

$$(10) \quad \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+1} f_n([x_i, x_j], x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_{n+1}) = l'_{n+1}(f_1(x_1), \dots, f_1(x_{n+1})).$$

#### 4. $L_\infty$ -ALGEBRAS FROM CLOSED DIFFERENTIAL FORMS

Here we recall various definitions and results about closed differential forms from previous work [25] [34], and introduce the ‘Poisson Lie  $n$ -algebra’ associated to any manifold equipped with a closed  $(n+1)$ -form.

**Definition 4.1.** An  $(n+1)$ -form  $\omega$  on a smooth manifold  $M$  is  **$n$ -plectic**, or more specifically an  **$n$ -plectic structure**, if it is both closed:

$$d\omega = 0,$$

and non-degenerate:

$$\forall x \in M \quad \forall v \in T_x M, \quad \iota_v \omega = 0 \Rightarrow v = 0.$$

If  $\omega$  is an  $n$ -plectic form on  $M$ , then we call the pair  $(M, \omega)$  an  **$n$ -plectic manifold**. More generally, if  $\omega$  is closed, but not necessarily non-degenerate, then we call  $(M, \omega)$  a **pre- $n$ -plectic manifold**

Obviously, a (pre-) 1-plectic manifold is a (pre-) symplectic manifold.



**Definition 4.2.** Let  $(M, \omega)$  be a pre- $n$ -plectic manifold. An  $(n-1)$ -form  $\alpha$  is **Hamiltonian** iff there exists a vector field  $v_\alpha \in \mathfrak{X}(M)$  such that

$$d\alpha = -\iota_{v_\alpha}\omega.$$

We say  $v_\alpha$  is a **Hamiltonian vector field** corresponding to  $\alpha$ . The set of Hamiltonian  $(n-1)$ -forms and the set of Hamiltonian vector fields on an pre- $n$ -plectic manifold are both vector spaces and are denoted as  $\Omega_{\text{Ham}}^{n-1}(M)$  and  $\mathfrak{X}_{\text{Ham}}(M)$ , respectively. Note that if  $\omega$  is  $n$ -plectic, then associated to every Hamiltonian form is a unique Hamiltonian vector field.

**Definition 4.3.** A vector field  $v$  on a pre- $n$ -plectic manifold  $(M, \omega)$  is a **local Hamiltonian vector field** iff

$$\mathcal{L}_v\omega = 0.$$

The set of local Hamiltonian vector fields is a vector space and is denoted as  $\mathfrak{X}_{\text{LHam}}(M)$ .

**Definition 4.4.** Let  $(M, \omega)$  be a pre- $n$ -plectic manifold. Given  $\alpha, \beta \in \Omega_{\text{Ham}}^{n-1}(M)$ , the **bracket**  $\{\alpha, \beta\}$  is the  $(n-1)$ -form given by

$$\{\alpha, \beta\} = \iota_{v_\beta}\iota_{v_\alpha}\omega,$$

where  $v_\alpha$  and  $v_\beta$  are any Hamiltonian vector fields for  $\alpha$  and  $\beta$  respectively.

The bracket is well-defined, for if both  $v_\alpha$  and  $v'_\alpha$  are Hamiltonian for  $\alpha \in \Omega_{\text{Ham}}^{n-1}(M)$ , then both  $\iota_{v_\beta}\iota_{v_\alpha}\omega$  and  $\iota_{v_\beta}\iota_{v'_\alpha}\omega$  are equal to  $-\iota_{v_\beta}d\alpha$ .

**Proposition 4.5.** *If  $(M, \omega)$  is a pre- $n$ -plectic manifold and  $v_1, v_2 \in \mathfrak{X}_{\text{LHam}}(M)$  are local Hamiltonian vector fields, then  $[v_1, v_2]$  is a global Hamiltonian vector field with*

$$d\iota(v_1 \wedge v_2)\omega = -\iota_{[v_1, v_2]}\omega,$$

and  $\mathfrak{X}_{\text{LHam}}(M)$  and  $\mathfrak{X}_{\text{Ham}}(M)$  are Lie subalgebras of  $\mathfrak{X}(M)$ .

*Proof.* if  $v_1, v_2$  are locally Hamiltonian, then by Eq. (5),

$$\mathcal{L}_{v_1}\iota_{v_2}\omega = \iota_{[v_1, v_2]}\omega.$$

On the other hand, by Eq. (4),

$$\mathcal{L}_{v_1}\iota_{v_2}\omega = \iota_{v_1}d\iota_{v_2}\omega + d\iota_{v_1}\iota_{v_2}\omega.$$

But  $\iota_{v_1}d\iota_{v_2}\omega = 0$ , since  $d\iota_{v_2} = \mathcal{L}_{v_2} - \iota_{v_2}d$ . □

Prop. 4.5 implies in particular that if  $v_\alpha$  and  $v_\beta$  are Hamiltonian vector fields for  $\alpha$  and  $\beta$  respectively, then  $[v_\alpha, v_\beta]$  is a Hamiltonian vector field for  $\{\alpha, \beta\}$ .

The next theorem gives a natural  $L_\infty$ -structure on differential forms, which extends the bracket  $\{\cdot, \cdot\}$  on  $\Omega_{\text{Ham}}^{n-1}(M)$ . The theorem is essentially Thm. 5.2 in [25], together with its generalization Thm. 6.7 in [34].

**Theorem 4.6.** *Given a pre- $n$ -plectic manifold  $(M, \omega)$ , there is a Lie  $n$ -algebra  $L_\infty(M, \omega) = (L, \{l_k\})$  with underlying graded vector space*

$$L^i = \begin{cases} \Omega_{\text{Ham}}^{n-1}(M) & i = 0, \\ \Omega^{n-1+i}(M) & 1 - n \leq i < 0, \end{cases}$$

and maps  $\{l_k: L^{\otimes k} \rightarrow L \mid 1 \leq k < \infty\}$  defined as

$$l_1(\alpha) = d\alpha,$$

if  $|\alpha| < 0$  and

$$l_k(\alpha_1, \dots, \alpha_k) = \begin{cases} \varsigma(k)\iota(v_{\alpha_1} \wedge \dots \wedge v_{\alpha_k})\omega & \text{if } |\alpha_1 \otimes \dots \otimes \alpha_k| = 0, \\ 0 & \text{if } |\alpha_1 \otimes \dots \otimes \alpha_k| < 0, \end{cases}$$

for  $k > 1$ , where  $v_{\alpha_i}$  is any Hamiltonian vector field associated to  $\alpha_i \in \Omega_{\text{Ham}}^{n-1}(M)$ .

**4.1. The Poisson Lie  $n$ -algebra.** Note that for the  $n = 1$  case, the underlying complex of  $L_\infty(M, \omega)$  is just the vector space of Hamiltonian functions  $C^\infty(M)_{\text{Ham}} \subseteq C^\infty(M)$ . The only non-trivial bracket is  $l_2 = \{\cdot, \cdot\}$ , which is a Lie bracket. Hence, we recover the the underlying Lie algebra of the usual Poisson algebra associated to a pre-symplectic manifold. In the symplectic case,  $C^\infty(M)_{\text{Ham}} = C^\infty(M)$ , and there is a well-defined surjective Lie algebra morphism

$$\pi: C^\infty(M) \rightarrow \mathfrak{X}_{\text{Ham}}(M)$$

sending a function to its unique Hamiltonian vector field. If  $M$  is connected, then we see  $\pi$  fits in the short exact sequence

$$(11) \quad 0 \rightarrow \mathbb{R} \rightarrow C^\infty(M) \xrightarrow{\pi} \mathfrak{X}_{\text{Ham}}(M) \rightarrow 0.$$

This is the Kostant-Souriau central extension [19, 30]. It characterizes the underlying Lie algebra of  $C^\infty(M)$ , up to isomorphism, as the unique central extension determined by the symplectic form (evaluated at a point  $p \in M$ ).

For the pre-symplectic case, Hamiltonian functions can have more than one corresponding Hamiltonian vector field, and so a map  $C^\infty(M)_{\text{Ham}} \rightarrow \mathfrak{X}_{\text{Ham}}(M)$  may not exist. Therefore one instead considers the Lie algebra

$$\begin{aligned} \widetilde{C^\infty(M)_{\text{Ham}}} &= \{(v, f) \in \mathfrak{X}_{\text{Ham}}(M) \oplus C^\infty(M)_{\text{Ham}} \mid df = -\iota_v \omega\} \\ [(v_1, f_1), (v_2, f_2)]_L &= ([v_1, v_2], \{f_1, f_2\}). \end{aligned}$$

The projection  $(v, f) \mapsto v$  then gives a central extension

$$(12) \quad 0 \rightarrow \mathbb{R} \rightarrow \widetilde{C^\infty(M)_{\text{Ham}}} \xrightarrow{\pi} \mathfrak{X}_{\text{Ham}}(M) \rightarrow 0$$

which generalizes (11) to any connected pre-symplectic manifold [5, Prop 2.3]. If  $(M, \omega)$  is symplectic, then it is easy to see that  $\widetilde{C^\infty(M)_{\text{Ham}}}$  is isomorphic to  $C^\infty(M)_{\text{Ham}} = C^\infty(M)$  as Lie algebras.

The higher analog of the central extension (12) for a pre- $n$ -plectic manifold is obtained by slightly modifying the construction of  $L_\infty(M, \omega)$ .

**Theorem 4.7.** *Given a pre- $n$ -plectic manifold  $(M, \omega)$ , there is a Lie  $n$ -algebra  $\text{PoissonLie}(M, \omega)$  with underlying graded vector space*

$$\begin{aligned} L^0 &= \widetilde{\Omega_{\text{Ham}}^{n-1}(M)} = \{(v, \alpha) \in \mathfrak{X}_{\text{Ham}}(M) \oplus \Omega_{\text{Ham}}^{n-1}(M) \mid d\alpha = -\iota_v \omega\} \\ L^i &= \Omega^{n-1+i}(M) \quad 1 - n \leq i < 0, \end{aligned}$$

and structure maps:

$$\begin{aligned} \tilde{l}_1(\alpha) &= \begin{cases} (0, d\alpha) & \text{if } |\alpha| = -1, \\ d\alpha & \text{if } |\alpha| < -1, \end{cases} \\ \tilde{l}_2(x_1, x_2) &= \begin{cases} ([v_1, v_2], \iota(v_1 \wedge v_2)\omega) = ([v_1, v_2], \{\alpha_1, \alpha_2\}) & \text{if } |x_1 \otimes x_2| = 0, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and, for  $k > 2$ :

$$\tilde{l}_k(x_1, \dots, x_k) = \begin{cases} \varsigma(k) \iota(v_{\alpha_1} \wedge \dots \wedge v_{\alpha_k})\omega & \text{if } |x_1 \otimes \dots \otimes x_k| < 0, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Eq. (6) is satisfied since the bracket of vector fields satisfies the Jacobi identity while the higher structure maps are identical to those of  $L_\infty(M, \omega)$ .  $\square$

We call  $\text{PoissonLie}(M, \omega)$  the **Poisson Lie  $n$ -algebra** associated to  $(M, \omega)$ .

**Proposition 4.8.** *Let  $(M, \omega)$  be a pre- $n$ -plectic manifold.*

(1) *The cochain map*

$$\pi: \text{PoissonLie}(M, \omega) \rightarrow \mathfrak{X}_{\text{Ham}}(M)$$

defined to be the projection  $(v, \alpha) \mapsto v$  in degree 0, and trivial in all lower degrees lifts to a strict morphism of  $L_\infty$ -algebras.

(2) *If  $(M, \omega)$  is  $n$ -plectic, then the cochain map*

$$\begin{array}{ccccccc} C^\infty(M) & \xrightarrow{d} & \Omega^1(M) & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega^{n-2}(M) & \xrightarrow{d} & \Omega_{\text{Ham}}^{n-1}(M) \\ \downarrow \text{id} & & \downarrow \text{id} & & & & \downarrow \text{id} & & \downarrow \phi \\ C^\infty(M) & \xrightarrow{d} & \Omega^1(M) & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega^{n-2}(M) & \xrightarrow{0 \oplus d} & \widetilde{\Omega_{\text{Ham}}^{n-1}(M)} \end{array}$$

with  $\phi(\alpha) = (v_\alpha, \alpha)$ , where  $v_\alpha$  is the unique Hamiltonian vector field associated to  $\alpha$ , lifts to a strict  $L_\infty$ -quasi-isomorphism.

*Proof.* To prove (1), note we have

$$\pi \tilde{l}_2((v_1, \alpha_1), (v_2, \alpha_2)) = [\pi(v_1, \alpha_1), \pi(v_2, \alpha_2)].$$

Hence, Eq. (7) is satisfied. For (2), note that we have the equalities

$$\begin{aligned} \tilde{l}_2 \phi^{\otimes 2} &= \phi l_2 \\ \tilde{l}_k \phi^{\otimes k} &= l_k \quad \forall k > 2. \end{aligned}$$

Hence, the cochain map lifts to a strict  $L_\infty$ -morphism. The map  $\phi$  is an isomorphism, hence the corresponding  $L_\infty$ -morphism is a quasi-isomorphism.  $\square$

**Remark 4.9.** We mention that there is a nice conceptual interpretation of the Lie  $n$ -algebra  $\text{PoissonLie}(M, \omega)$  within the context of differential cohomology whenever  $\omega$  represents an integral cohomology class i.e.

$$[\omega] \in \text{im}(H^{n+1}(M, \mathbb{Z}) \rightarrow H^{n+1}(M, \mathbb{R})).$$

For any manifold  $M$  there is a short exact sequence of groups

$$0 \rightarrow H^n(M, \text{U}(1)) \rightarrow H_{\text{Del}}^n(M) \xrightarrow{\text{curv}} \Omega_{\text{cl/int}}^{n+1}(M) \rightarrow 0,$$

where  $H^\bullet(M, \text{U}(1))$  is ordinary  $\text{U}(1)$ -valued cohomology,  $\Omega_{\text{cl/int}}^{n+1}(M)$  is the group of closed and integral  $(n+1)$ -forms, and  $H_{\text{Del}}^\bullet(M)$  is ‘smooth Deligne cohomology’ [5, Sec. 1.5]. The group  $H_{\text{Del}}^n(M)$  classifies certain higher geometric objects which one could call ‘principal  $\text{U}(1)$   $n$ -bundles’ equipped with an ‘ $n$ -connection’. The curvature of such an  $n$ -connection is given by the surjection in the above sequence, and therefore is an integral pre- $n$ -plectic form on  $M$ .

For example,  $H_{\text{Del}}^1(M)$  is in bijection with isomorphism classes of principal  $\text{U}(1)$  bundles with connection over  $M$ . The surjection in the above sequence sends such a bundle to its curvature 2-form. The group  $H_{\text{Del}}^2(M)$  classifies geometric objects called  $\text{U}(1)$ -gerbes equipped a ‘2-connection’ (also called a ‘connective structure’ and its ‘curving’), whose curvature is a closed integral 3-form on  $M$ .

Kostant [19] showed that the Lie algebra of infinitesimal symmetries of a principal  $\text{U}(1)$ -bundle which preserve a fixed connection is isomorphic to the Lie algebra that we call  $\text{PoissonLie}(M, \omega)$ , when  $(M, \omega)$  is pre-symplectic. This fact plays an essential role in geometric quantization.

It is known that the infinitesimal symmetries of gerbes form a Lie 2-algebra (for example, see [9]), and this pattern continues for  $n > 2$ . Indeed, in [12] it is shown for any  $n > 1$  that the Lie  $n$ -algebra of connection-preserving infinitesimal symmetries of a principal  $\text{U}(1)$   $n$ -bundle whose curvature is  $\omega$  is quasi-isomorphic to the Lie  $n$ -algebra  $\text{PoissonLie}(M, \omega)$ .

## 5. HOMOTOPY MOMENT MAPS

In symplectic geometry, a moment map  $M \rightarrow \mathfrak{g}^*$  can be equivalently expressed as a ‘co-moment map’ i.e. a Lie algebra morphism  $\mathfrak{g} \rightarrow C^\infty(M)$ . In this section, we describe the natural  $L_\infty$  analog of this co-moment map. We call this a ‘homotopy moment map’.

**Definition/Proposition 5.1.** *Let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$ . Let  $(M, \omega)$  be a pre- $n$ -plectic manifold equipped with a  $G$  action which preserves  $\omega$ , and such that the infinitesimal  $\mathfrak{g}$  action  $x \mapsto v_x$  is via Hamiltonian vector fields. A **homotopy moment map** (or **moment map** for short) is a lift*

$$(13) \quad \begin{array}{ccc} & \text{PoissonLie}(M, \omega) & \\ & \downarrow \pi & \\ \mathfrak{g} & \xrightarrow{v_-} & \mathfrak{X}_{\text{Ham}}(M). \end{array}$$

of the Lie algebra morphism  $v_-$  (1) through the  $L_\infty$ -morphism  $\pi$  (1) in the category of  $L_\infty$ -algebras. Such a lift corresponds to an  $L_\infty$ -morphism

$$(f_k): \mathfrak{g} \rightarrow L_\infty(M, \omega)$$

such that

$$-\iota_{v_x} \omega = d(f_1(x)) \quad \text{for all } x \in \mathfrak{g}.$$

Before we give a proof of the above correspondence between lifts and morphisms into  $L_\infty(M, \omega)$ , we explain in more detail what a homotopy moment map actually is:

- The condition  $-\iota_{v_x} \omega = d(f_1(x))$  implies that  $v_x$  is a Hamiltonian vector field for  $f_1(x) \in \Omega_{\text{Ham}}^{n-1}(M)$ .
- By Prop. 3.8, an  $L_\infty$ -morphism  $\mathfrak{g} \rightarrow L_\infty(M, \omega)$  consists of a collection of  $n$  skew-symmetric maps

$$f_k: \mathfrak{g}^{\otimes k} \rightarrow L, \quad 1 \leq k \leq n,$$

where  $L$  is the underlying vector space of  $L_\infty(M, \omega)$  and  $|f_k| = 1 - k$ , which satisfy

$$(14) \quad \sum_{1 \leq i < j \leq k} (-1)^{i+j+1} f_{k-1}([x_i, x_j], x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_k) \\ = df_k(x_1, \dots, x_k) + \varsigma(k) \iota(v_1 \wedge \dots \wedge v_k) \omega$$

for  $2 \leq k \leq n$  and

$$(15) \quad \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+1} f_n([x_i, x_j], x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_{n+1}) = \varsigma(n+1) \iota(v_1 \wedge \dots \wedge v_{n+1}) \omega,$$

where  $v_i$  is the vector field associated to  $x_i$  via the  $\mathfrak{g}$ -action. (These equalities are obtained from Eqs. (9) and (10) via substitution using the definition of  $f_1$  and the maps  $l_k$  defined in Thm. 4.6. Notice that Thm. 4.6 implies that  $L_\infty(M, \omega)$  has property (P).)

- Finally, note that Prop. 4.5 imply that  $v_{[x, y]} = [v_x, v_y]$  is a Hamiltonian vector field for

$$\{f_1(x), f_1(y)\} = l_2(f_1(x), f_1(y)).$$

Of course,  $f_1: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^{n-1}(M)$  need not preserve the bracket on  $\mathfrak{g}$  i.e. in general  $f_1([x, y]) \neq \{f_1(x), f_1(y)\}$ . This is a good property, in view of the facts that the Lie bracket of  $\mathfrak{g}$  satisfies the Jacobi identity but  $\{\cdot, \cdot\}$  does not.

*Proof of Def./Prop. 5.1.* Suppose we have an  $L_\infty$ -morphism

$$(\tilde{f}_k): \mathfrak{g} \rightarrow \text{PoissonLie}(M, \omega)$$

corresponding to a lift (13). Note that  $\text{PoissonLie}(M, \omega)$  satisfies Property (P). Therefore the morphisms  $(\tilde{f}_k)$  are trivial for  $k \geq n+1$  and satisfy the compatibility equations given in Prop. 3.8. By the definition of the projection  $\pi$ , the degree 0 map can be written as

$$\tilde{f}_1(x) = (v_x, f_1(x)) \in \widetilde{\Omega_{\text{Ham}}^{n-1}(M)}$$

where  $f_1: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^{n-1}(M)$  is a linear map satisfying  $-\iota_{v_x}\omega = d(f_1(x))$ . Moreover,  $\pi$  is a strict  $L_\infty$ -morphism and therefore

$$\tilde{f}_1([x, y]) = (v_{[x, y]}, f_1([x, y])) = ([v_x, v_y], f_1([x, y])).$$

Combining these observations with the fact that structure maps  $\tilde{l}_k$  of  $\text{PoissonLie}(M, \omega)$  agree with those of  $L_\infty(M, \omega)$  for  $k \geq 3$ , we obtain an  $L_\infty$  morphism

$$(f_k): \mathfrak{g} \rightarrow L_\infty(M, \omega)$$

with  $f_k = \tilde{f}_k$  for  $k > 1$ . The fact that every such morphism gives a lift is now obvious.  $\square$

**Remark 5.2.** Note we have a generalization of the fact that, in symplectic geometry, the image of a moment map is a Lie subalgebra of the Poisson algebra of functions on the symplectic manifold.

More precisely, given a moment map with components  $f_k: \mathfrak{g}^{\otimes k} \rightarrow L$  (for  $1 \leq k \leq n$ ), its image is not an  $L_\infty$ -subalgebra of  $L_\infty(M, \omega)$  in general (unless  $n = 1$ ). However, the subcomplex generated by its image, which we denote by  $I$  and which is given by

$$I^k = \begin{cases} \text{im}(f_n) & k = -n + 1 \\ \text{im}(f_{-k+1}) + d(\text{im}(f_{-k+2})) & -n + 2 \leq k \leq 0 \end{cases}$$

is an  $L_\infty$ -subalgebra of  $L_\infty(M, \omega)$ . Indeed  $I$  is closed w.r.t. the differential  $d$  by construction. It is seen to be closed w.r.t. the higher brackets using the definition of the latter (Thm. 4.6) together with Eq. (14) and (15), and because the Hamiltonian vector field of an exact element of  $\Omega_{\text{Ham}}^{n-1}(M)$  is zero.

## 6. EQUIVARIANT DE RHAM COHOMOLOGY

In this section we establish that moment maps can be constructed out of certain equivariant cocycles (see Thm. 6.3), generalizing a known fact in symplectic geometry.

Let  $G$  be a compact Lie group acting on  $M$ . As in Sec. 5, we denote the induced infinitesimal action by  $\mathfrak{g} \rightarrow \mathfrak{X}(M), x \mapsto v_x$ . Furthermore, there is the corresponding  $G$  action on differential forms on  $M$ , and, of course,  $G$  acts on  $\mathfrak{g}^*$  via the coadjoint action.

**6.1. Extensions in the Cartan model.** Recall that the Cartan model for equivariant de Rham cohomology of  $M$  [17, Sec. 4.2] is the complex

$$C_G(M) = (\Omega^\bullet(M) \otimes S^\bullet(\mathfrak{g}^*[-2]))^G$$

with differential

$$d_G(\alpha)(x) = d(\alpha(x)) - \iota_{v_x}(\alpha(x)) \quad \forall x \in \mathfrak{g},$$

where  $\alpha \in C_G(M)$  is regarded as a polynomial map  $\alpha: \mathfrak{g} \rightarrow \Omega^\bullet(M)$ . For any  $k$ , denote by

$$C_G^k(M) = \oplus_{2j \leq k} \Omega^{k-2j}(M) \otimes S^j(\mathfrak{g}^*[-2])$$

the degree  $k$  component of the Cartan model  $C_G(M)$ . Given an element  $\tilde{\omega} \in C_G^k(M)$ , we denote by  $\tilde{\omega}_j$  its component in  $\Omega^{k-2j}(M) \otimes S^j(\mathfrak{g}^*[-2])$ , for all  $j = 1, \dots, [\frac{k}{2}]$ . Hence,

$$\tilde{\omega} = \tilde{\omega}_0 + \dots + \tilde{\omega}_{[\frac{k}{2}]}.$$

**Definition 6.1.** An **extension** of an invariant closed differential form  $\omega \in \Omega^k(M)$  is a cocycle  $\tilde{\omega} \in C_G^k(M)$  such that

$$\tilde{\omega}_0 = \omega.$$

A  **$j$ -step extension** is an extension of the form

$$\tilde{\omega} = \tilde{\omega}_0 + \cdots + \tilde{\omega}_j.$$

Let  $G$  be a connected Lie group acting on the pre- $n$ -plectic manifold  $(M, \omega)$ . We will be interested 1-step extensions of  $\omega$ , i.e. in degree  $n+1$  elements of the form  $\omega - \mu \in C_G(M)$  with

$$\omega \in \Omega^{n+1}(M), \quad \mu \in \Omega^{n-1}(M) \otimes \mathfrak{g}^*[-2].$$

Observe that invariance of  $\omega - \mu$  under the  $G$  action is equivalent to

$$(16) \quad \mathcal{L}_{v_x} \omega = 0, \quad \mathcal{L}_{v_x} \mu(y) = \mu([x, y])$$

for all  $x, y \in \mathfrak{g}$ .

Further, the element  $\omega - \mu \in C_G(M)$  is closed w.r.t.  $d_G$  iff it satisfies

$$(17) \quad d\omega = 0, \quad d\mu(x) = -\iota_{v_x} \omega, \quad \iota_{v_x} \mu(x) = 0$$

for all  $x \in \mathfrak{g}$ .

The following proposition will be very useful for proving the existence of homotopy moment maps:

**Proposition 6.2.** *If  $\omega - \mu \in C_G(M)$  is a 1-step extension of  $\omega$ , then for  $k = 1, \dots, n$  the linear map*

$$(18) \quad \begin{aligned} f_k &: \mathfrak{g}^{\otimes k} \rightarrow \Omega^{n-k}(M) \\ f_k(x_1, \dots, x_k) &= \varsigma(k) \iota(v_{x_1} \wedge \cdots \wedge v_{x_{k-1}}) \mu(x_k) \end{aligned}$$

*is skew-symmetric and satisfies*

$$(19) \quad \mathcal{L}_{v_{x_1}}(f_k(x_2, \dots, x_{k+1})) = \sum_{i=2}^{k+1} (-1)^i f_k([x_1, x_i], x_2, \dots, \widehat{x_i}, \dots, x_{k+1})$$

*Proof.* First, we show skew-symmetry. The last equality in Eq. (17) implies that

$$\iota_{v_x} \mu(y) = -\iota_{v_y} \mu(x) \quad \forall x, y \in \mathfrak{g},$$

which further implies

$$\begin{aligned} f_k(x_1, \dots, x_k) &= \varsigma(k) \iota_{v_{k-1}} \iota_{v_{k-2}} \cdots \iota_{v_1} \mu(x_k) \\ &= -\varsigma(k) \iota_{v_{k-1}} \iota_{v_{k-2}} \cdots \iota_{v_k} \mu(x_1), \end{aligned}$$

where we write  $v_i$  for the vector field  $v_{x_i}$ . Since  $\mathfrak{g}$  is concentrated in deg 0, the above implies  $f_k$  is graded skew-symmetric.

Next we prove Eq. (19) via induction on  $k$ . The last equality in Eq. (16) gives the base case  $\mathcal{L}_{v_1} f_1(x_2) = f_1([x_1, x_2])$ .

Now assume Eq. (19) holds for  $k \geq 1$ . We have

$$\begin{aligned} f_{k+1}(x_2, \dots, x_{k+1}, x_{k+2}) &= -f_{k+1}(x_2, \dots, x_{k+2}, x_{k+1}) \\ &= -\varsigma(k+1) \iota(v_2 \wedge \cdots \wedge v_k \wedge v_{k+2}) \mu(x_{k+1}) \\ &= -\varsigma(k+1) \iota_{v_{k+2}} \iota(v_2 \wedge \cdots \wedge v_k) \mu(x_{k+1}) \\ &= -\frac{\varsigma(k+1)}{\varsigma(k)} \iota_{v_{k+2}} f_k(x_2, \dots, x_{k+1}), \end{aligned}$$



using the skew-symmetry of  $f_{k+1}$  in the first equation. Therefore, Eq. (5) gives

$$(20) \quad \begin{aligned} \mathcal{L}_{v_1} f_{k+1}(x_2, \dots, x_{k+1}, x_{k+2}) = \\ - \frac{\varsigma(k+1)}{\varsigma(k)} (\iota_{[v_1, v_{k+2}]} f_k(x_2, \dots, x_{k+1}) + \iota_{v_{k+2}} \mathcal{L}_{v_1} f_k(x_2, \dots, x_{k+1})). \end{aligned}$$

The first term on the right hand side of Eq. (20) is

$$\begin{aligned} - \frac{\varsigma(k+1)}{\varsigma(k)} \iota_{[v_1, v_{k+2}]} f_k(x_2, \dots, x_{k+1}) &= -\varsigma(k+1) \iota_{[v_1, v_{k+2}]} \iota(v_2 \wedge \dots \wedge v_k) \mu(x_{k+1}) \\ &= -\varsigma(k+1) \iota(v_2 \wedge \dots \wedge v_k \wedge [v_1, v_{k+2}]) \mu(x_{k+1}) \\ &= -(-1)^{k-1} \varsigma(k+1) \iota([v_1, v_{k+2}] \wedge v_2 \wedge \dots \wedge v_k) \mu(x_{k+1}) \\ &= (-1)^{k+2} \varsigma(k+1) \iota([v_1, v_{k+2}] \wedge v_2 \wedge \dots \wedge v_k) \mu(x_{k+1}) \\ &= (-1)^{k+2} f_{k+1}([x_1, x_{k+2}], \dots, x_{k+1}). \end{aligned}$$

The second term on the r.h.s. of Eq. (20) is

$$\begin{aligned} - \frac{\varsigma(k+1)}{\varsigma(k)} \iota_{v_{k+2}} \mathcal{L}_{v_1} f_k(x_2, \dots, x_{k+1}) &= -\varsigma(k+1) \sum_{i=2}^k (-1)^i \iota_{v_{k+2}} \iota([v_1, v_i] \wedge v_2 \wedge \dots \wedge \widehat{v_i} \wedge \dots \wedge v_k) \mu(x_{k+1}) \\ &\quad - \varsigma(k+1) (-1)^{k+1} \iota_{v_{k+2}} \iota([v_1, v_{k+1}] \wedge v_2 \wedge \dots \wedge v_{k-1}) \mu(x_k) \\ &= - \sum_{i=2}^k (-1)^i f_{k+1}([x_1, x_i], \dots, \widehat{x_i}, \dots, x_{k+2}, x_{k+1}) \\ &\quad - (-1)^{k+1} f_{k+1}([x_1, x_{k+1}], \dots, x_{k+2}, x_k) \\ &= \sum_{i=2}^{k+1} (-1)^i f_{k+1}([x_1, x_i], \dots, \widehat{x_i}, \dots, x_{k+1}, x_{k+2}), \end{aligned}$$

where in the first equality we used the induction hypothesis, in the second the definition of  $f_{k+1}$ , and in the third its skew-symmetry.

Therefore Eq. (20) gives the desired result for  $k+1$ .  $\square$

**6.2. Moment maps from 1-step extensions.** The following is one of the main results of our paper.

**Theorem 6.3.** *If  $(M, \omega)$  is a pre- $n$ -plectic manifold equipped with a  $G$ -action, and  $\omega - \mu$  is a 1-step extension of  $\omega$ , then the maps*

$$\begin{aligned} f_k &: \mathfrak{g}^{\otimes k} \rightarrow \Omega^{n-k}(M), \\ f_k(x_1, \dots, x_k) &= \varsigma(k) \iota(v_{x_1} \wedge \dots \wedge v_{x_{k-1}}) \mu(x_k), \end{aligned}$$

for  $1 \leq k \leq n$ , are the components of a  $G$ -equivariant moment map  $\mathfrak{g} \rightarrow L_\infty(M, \omega)$ .

*Proof.* The maps  $f_k$  are skew-symmetric by Prop. 6.2. Since  $d_G(\omega - \mu) = 0$ , the equality  $d\mu(x) = -\iota_{v_x} \omega$  holds for all  $x \in \mathfrak{g}$ .

Notice that in particular the image of  $f_1$  is contained in  $\Omega_{\text{Ham}}^{n-1}(M)$ , and the image of each  $f_k$  lies in the degree  $k-1$  component of  $L_\infty(M, \omega)$ . By Prop. 3.8, to show that the  $f_k$  are the components of an  $L_\infty$ -morphism, we just have to show that Eq. (14) and Eq. (15) hold.

We first prove Eq. (14) holds. We proceed via induction on  $k$ . For  $k = 2$ , the last equality in Eq. (16) gives  $\mathcal{L}_{v_1} f_1(x_2) = f_1([x_1, x_2])$ . On the other hand,

$$\begin{aligned}\mathcal{L}_{v_1} f_1(x_2) &= \iota_{v_1} d\mu(x_2) + d\iota_{v_1} \mu(x_2) \\ &= \iota_{v_2} \iota_{v_1} \omega + d\iota_{v_1} \mu(x_2) \\ &= df_2(x_1, x_2) + \varsigma(2) \iota(v_1 \wedge v_2) \omega\end{aligned}$$

where we used  $\varsigma(1) = \varsigma(2) = 1$ . Hence, Eq. (14) holds.

Now assume Eq. (14) holds for  $2 < k < n$ . Substituting  $f_k$  into Eq. (19) from Prop. 6.2 gives

$$(21) \quad \sum_{i=2}^{k+1} (-1)^i f_k([x_1, x_i], x_2, \dots, \widehat{x_i}, \dots, x_{k+1}) \\ = \iota_{v_1} df_k(x_2, \dots, x_{k+1}) + d\iota_{v_1} f_k(x_2, \dots, x_{k+1}).$$

The second term on the r.h.s. of Eq. (21) can be re-written using

$$(22) \quad \begin{aligned}\iota_{v_1} f_k(x_2, \dots, x_{k+1}) &= \iota_{v_1} \varsigma(k) \iota(v_2 \wedge \dots \wedge v_k) \mu(x_{k+1}) \\ &= \varsigma(k) \iota(v_2 \wedge \dots \wedge v_k \wedge v_1) \mu(x_{k+1}) = (-1)^{k-1} \frac{\varsigma(k)}{\varsigma(k+1)} f_{k+1}(x_1, \dots, x_{k+1})\end{aligned}$$

For the first term on the r.h.s. of Eq. (21) we have

$$(23) \quad \begin{aligned}\iota_{v_1} df_k(x_2, \dots, x_{k+1}) &= \sum_{2 \leq i < j \leq k+1} (-1)^{i+j+1} \iota_{v_1} f_{k-1}([x_i, x_j], x_2, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_{k+1}) \\ &\quad - \varsigma(k) \iota(v_2 \wedge \dots \wedge v_{k+1} \wedge v_1) \omega \\ &= \sum_{2 \leq i < j \leq k+1} (-1)^{i+j+1} \frac{\varsigma(k-1)}{\varsigma(k)} (-1)^{k-1} f_k([x_i, x_j], x_1, x_2, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_{k+1}) \\ &\quad - (-1)^k \varsigma(k) \iota(v_1 \wedge v_2 \wedge \dots \wedge v_{k+1}) \omega,\end{aligned}$$

where in the first equation we used the induction hypothesis

$$\begin{aligned}\sum_{2 \leq i < j \leq k+1} (-1)^{i+j+1} f_{k-1}([x_i, x_j], x_2, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_{k+1}) \\ = df_k(x_2, \dots, x_{k+1}) + \varsigma(k) \iota(v_2 \wedge \dots \wedge v_{k+1}) \omega\end{aligned}$$

and in the second we used a computation analog to Eq. (22).

Substituting Eq. (23) and Eq. (22) into Eq. (21) and rearranging terms, we see that Eq. (21) is equivalent to:

$$(24) \quad \begin{aligned}\sum_{j=2}^{k+1} (-1)^{1+j+1} f_k([x_1, x_j], x_2, \dots, \widehat{x_j}, \dots, x_{k+1}) \\ + (-1)^k \frac{\varsigma(k-1)}{\varsigma(k)} \sum_{2 \leq i < j \leq k+1} (-1)^{i+j+1} f_k([x_i, x_j], x_1, x_2, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_{k+1}) \\ = (-1)^{k-1} \frac{\varsigma(k)}{\varsigma(k+1)} \left( df_{k+1}(x_1, \dots, x_{k+1}) + \varsigma(k+1) \iota(v_1 \wedge v_2 \wedge \dots \wedge v_{k+1}) \omega \right).\end{aligned}$$

Applying  $\varsigma(m-1)\varsigma(m) = (-1)^m$  to  $m = k$  and  $m = k+1$  we deduce that Eq. (14) holds for  $k+1 \leq n$ .

It is not difficult to see that Eq. (15) holds via the same proof as above, i.e. by substituting  $f_n$  into Eq. (19), while noting that

$$d\iota(v_1 \wedge \cdots \wedge v_n)\mu(x_{n+1}) = 0$$

since  $\mu(x_{n+1}) \in \Omega^{n-1}(M)$ , and then using Eq. (14) with  $k = n$ . This concludes the proof that the  $f_k$  are the components of an  $L_\infty$ -morphism.

The equivariance of  $f_k: \mathfrak{g}^{\otimes k} \rightarrow \Omega^{n-k}(M)$  means that for all  $x, x_1, \dots, x_k \in \mathfrak{g}$  we have

$$\sum_{j=1}^k f_k(x_1, \dots, [x, x_j], \dots, x_k) = \mathcal{L}_{v_x}(f_k(x_1, \dots, x_k)).$$

Expressing  $f_k$  in terms of  $\mu$ , the equivariance of  $f_k$  is expressed as

$$\iota([v_x, v_{x_1} \wedge \cdots \wedge v_{x_{k-1}}])\mu(x_k) + \iota(v_{x_1} \wedge \cdots \wedge v_{x_{k-1}})\mu([x, x_k]) = \mathcal{L}_{v_x}(\iota(v_{x_1} \wedge \cdots \wedge v_{x_{k-1}})\mu(x_k)),$$

where we used Eq. (2). This equation is always satisfied, as one sees applying Eq. (5) to  $\mu(x_k)$  and using the  $G$ -equivariance of  $\mu$  (Eq. (16)).  $\square$

**Remark 6.4.** In the above proof, only the algebraic properties of the complex  $C_G(M)$  representing the Cartan model are used to show that 1-step extensions give equivariant moment maps. In particular, we did not need to use the fact that the cohomology of  $C_G(M)$  is isomorphic to the  $G$ -equivariant real cohomology of  $M$ . Hence, Thm. 6.3 also implies that we can produce equivariant moment maps from 1-step extensions in  $C_G(M)$  where  $G$  is any *non-compact* Lie group as well. Of course, in this case, we are no longer necessarily within the realm of equivariant cohomology; however, this does provide a useful algebraic tool to build examples with.

**6.3. More general cocycles.** As mentioned in the introduction, we expect any arbitrary extension of  $\omega$  in the Cartan model to induce a moment map, thus extending the result of Thm. 6.3 for 1-step extensions. We display a situation in which 2-step extensions arise naturally.

Let  $G_i$  act on the manifold  $M_i$  and  $\alpha_i$  be an equivariant cocycle in the Cartan model for this action, i.e.  $d_{G_i}\alpha_i = 0$  ( $i = 1, 2$ ). Consider the product  $\alpha_1\alpha_2$  (obtained simply wedge-multiplying the differential form components). Then  $\alpha_1\alpha_2$  is an equivariant cocycle in the Cartan model for the product action of  $G_1 \times G_2$  on  $M_1 \times M_2$ , since

$$d_{G_1 \times G_2}(\alpha_1\alpha_2) = (d_{G_1 \times G_2}\alpha_1)\alpha_2 \pm \alpha_1(d_{G_1 \times G_2}\alpha_2) = (d_{G_1}\alpha_1)\alpha_2 \pm \alpha_1(d_{G_2}\alpha_2) = 0.$$

We spell this out when the equivariant cocycles are of the kind considered in Thm. 6.3:

**Proposition 6.5.** *Let  $G_i$  act on  $(M_i, \omega_i)$  with  $\omega_i \in \Omega^{n_i+1}(M_i)$  a closed form, for  $i = 1, 2$ . Let  $\omega_i - \mu_i$  be 1-step extensions, and regard  $\mu_i$  as maps  $\mathfrak{g}_i \rightarrow \Omega^{n_i-1}(M_i)$ . Then the product action of  $G_1 \times G_2$  on the pre- $(n_1 + n_2 + 1)$ -plectic manifold  $(M_1 \times M_2, \omega_1\omega_2)$  admits an equivariant extension given by*

$$\omega_1\omega_2 - \eta + p$$

where

$$\begin{aligned} \eta: \mathfrak{g}_1 \oplus \mathfrak{g}_2 &\rightarrow \Omega^{n_1+n_2}(M_1 \times M_2), & x_1 + x_2 &\mapsto \mu_1^{x_1}\omega_2 + \omega_1\mu_2^{x_2} \\ p: S^2(\mathfrak{g}_1 \oplus \mathfrak{g}_2) &\rightarrow \Omega^{n_1+n_2-2}(M_1 \times M_2), & (x_1 + x_2, y_1 + y_2) &\mapsto \frac{1}{2}(\mu_1^{x_1}\mu_2^{y_2} + \mu_1^{y_1}\mu_2^{x_2}). \end{aligned}$$

(Here we denote by  $x_i, y_i, \dots$  elements of  $\mathfrak{g}_i$ , by  $v_{x_i}$  the corresponding vector fields on  $M_i$ , and for the sake of readability we omit wedge products and write  $\mu^x$  for  $\mu(x)$ .)

In particular, when  $\omega_1$  and  $\omega_2$  are symplectic, the action of  $G_1 \times G_2$  on the 3-plectic manifold  $(M_1 \times M_2, \omega_1\omega_2)$  admits a moment map.

## 7. CLOSED 3-FORMS

In this section, we analyze the simplest case in which non-strict moment maps appear, namely the case of closed 3-forms. Under the assumption that the Lie group  $G$  is compact semisimple, we prove the existence of moment maps provided the  $G$ -action has a fixed point, and the uniqueness of moment maps up to a certain equivalence. Furthermore, in the general setting, we show that not all equivariant moment maps arise from 1-step extensions.

An important example is the Lie group itself equipped with its Cartan 3-form. We consider this as a special case later on in Sec. 8.3.

**7.1. Review of symplectic case.** Let us first briefly discuss the case of 2-forms. In this case, moment maps as in Def. 5 are necessarily strict by degree reasons. Let  $G$  be a Lie group acting on a symplectic manifold  $(M, \omega)$ . The classical notion of equivariant moment map is the following [7]: a map  $J: M \rightarrow \mathfrak{g}^*$  such that  $v_x$  is the Hamiltonian vector field of  $J^*(x)$  for all  $x \in \mathfrak{g}$  and so that  $J$  is equivariant w.r.t. the  $G$ -action on  $M$  and the coadjoint action of  $G$  on  $\mathfrak{g}^*$ . In terms of the co-moment map, i.e. the pullback of functions  $f = J^*: \mathfrak{g} \rightarrow C^\infty(M)$ , this means

$$\begin{aligned} v_x &\text{ is the Hamiltonian vector field of } f(x), \text{ for all } x \in \mathfrak{g}, \\ f: (\mathfrak{g}, [\cdot, \cdot]) &\rightarrow (C^\infty(M), \{\cdot, \cdot\}) \text{ is a Lie algebra morphism,} \end{aligned}$$

where  $\{\cdot, \cdot\}$  the Poisson bracket on  $M$ . Hence, in the symplectic case, our Def. 5.1 agrees with classical notion of moment map.

An equivalent characterization of moment map is that  $\omega - f$  is a closed degree 2 element of the Cartan model for equivariant cohomology. In other words: in the symplectic case, all homotopy moment maps arise from equivariant extensions of  $\omega$ .

**7.2. Notation.** In the remainder of this section, let  $\omega$  be a closed 3-form on  $M$  which is invariant under the action of a Lie group  $G$ . In this case, a moment map (Def. 5.1) consists of two components

$$f_1: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^1(M), \quad f_2: \mathfrak{g} \otimes \mathfrak{g} \rightarrow C^\infty(M),$$

where the second is skew-symmetric, such that

A.  $v_x$  is the Hamiltonian vector field of  $f_1(x)$ , for all  $x \in \mathfrak{g}$ ,

B. the following two equations are satisfied:

$$(25) \quad f_1([x, y]) - \underbrace{\{f_1(x), f_1(y)\}}_{\omega(v_x, v_y, \cdot)} = df_2(x, y)$$

$$(26) \quad -\underbrace{l_3(f_1(x), f_1(y), f_1(z))}_{\omega(v_x, v_y, v_z)} = f_2(x, [y, z]) - f_2(y, [x, z]) + f_2(z, [x, y]).$$

**7.3. Existence of moment maps.** If  $G$  is also connected and semisimple, then it is well-known for the symplectic case that a symplectic  $G$ -action admits a moment map [7, Chapter X]. However, for the present case, we now need an additional condition on the action.

**Proposition 7.1.** *If  $G$  is compact, connected, and semisimple and for all  $x \in \mathfrak{g}$  there is point  $p \in M$  such that  $v_x(p) = 0$ , then there exists an equivariant homotopy moment map.*

Notice that the assumptions on the action in Prop. 7.1 is satisfied when  $M$  is oriented, compact, and has non-zero Euler characteristic; since in this case the Poincaré-Hopf theorem implies that every vector field on  $M$  has a zero.

The proof of Prop. 7.1 is based on the following lemmas, which are straightforward analogs of well-known statements in symplectic geometry. We phrase these more generally for  $\omega$  any invariant closed form  $\omega \in \Omega^{n+1}(M)$  with  $n \geq 2$ .

**Lemma 7.2.** *If  $\mathfrak{g}$  satisfies  $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ , then there exists a linear map  $\mu: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^{n-1}(M)$  such that  $v_x$  is a Hamiltonian vector field for  $\mu(x)$ , that is,  $d\mu(x) = -\iota_{v_x}\omega$  for all  $x \in \mathfrak{g}$ .*

*Proof.* Let  $x \in \mathfrak{g}$ . Since  $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ , we can write  $x = \sum_i [x_i, x'_i]$ . The locally Hamiltonian vector field  $v_x$  can hence be written as  $v_{\sum_i [x_i, x'_i]} = \sum_i [v_{x_i}, v_{x'_i}]$ , and by Prop. 4.5 it is the Hamiltonian vector field of

$$\mu(x) = \sum_i \iota(v_{x_i} \wedge v_{x'_i})\omega.$$

Now let  $x$  range through a basis of  $\mathfrak{g}$ , and extend  $\mu$  to obtain a linear map  $\mu: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^{n-1}(M)$ .  $\square$

**Lemma 7.3.** *If  $G$  is compact and the linear map  $\mu: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^{n-1}(M)$  satisfies  $d\mu(x) = -\iota_{v_x}\omega$  for all  $x \in \mathfrak{g}$ , then*

$$\mu'(x) = \int_G g^*(\mu(\text{Ad}_g x))$$

*is equivariant and satisfies  $d\mu'(x) = -\iota_{v_x}\omega$ .*

*Proof.* For every  $g \in G$  and  $x \in \mathfrak{g}$ , one computes that  $d(g^*(\mu(\text{Ad}_g x))) = -\iota_{v_x}\omega$  using the fact that  $\omega$  is  $G$ -invariant. Integrating over  $G$  we obtain a map  $\mu'$  which is equivariant.  $\square$

**Lemma 7.4.** *If the linear map  $\mu: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^{n-1}(M)$  is equivariant and  $d\mu(x) = -\iota_{v_x}\omega$ , then  $\iota_{v_x}\mu(x)$  is automatically a closed  $n-2$  form for all  $x \in \mathfrak{g}$ .*

*Proof.* The equivariance of  $\mu$  is equivalent to  $\mathcal{L}_{v_x}\mu(y) = \mu([x, y])$  for all  $x, y \in \mathfrak{g}$ . Now  $d\iota_{v_x}\mu(x) = -\iota_{v_x}d\mu(x) + \mathcal{L}_{v_x}\mu(x) = 0$ .  $\square$

*Proof of Prop. 7.1.* Since  $G$  is semisimple, Lemma 7.2 produces  $\mu: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^{n-1}(M)$  satisfying  $d\mu(x) = -\iota_{v_x}\omega$ . The compactness of  $G$  allows to apply Lemma 7.3, and therefore we may assume that  $\mu$  is equivariant. For every  $x \in \mathfrak{g}$ , the function  $\iota_{v_x}\mu(x)$  is a constant function by Lemma 7.4. We conclude that it must be identically zero since it vanishes at  $p$ . Hence  $\omega - \mu$  is a 1-step extension, so Thm. 6.3 produces an equivariant moment map with components  $f_1 = \mu$  and  $f_2(x, y) = \iota_{v_x}\mu(y)$ .  $\square$

**7.4. Uniqueness of moment maps.** Next, we comment briefly on uniqueness issues. Recall that a moment map in symplectic geometry is unique if  $H^1(\mathfrak{g}, \mathbb{R}) = 0$  [7, Section 26]. Similarly, in the pre-2-plectic case, cohomological constraints on  $\mathfrak{g}$  will ensure uniqueness, but only up to a certain equivalence given by an action of  $C^\infty(M)$ .

**Lemma 7.5.**

- a) *If  $\omega - \mu$  is a 1-step extension of  $\omega$  and  $\psi: \mathfrak{g} \rightarrow C^\infty(M)$  is any  $G$ -equivariant linear map, then  $\omega - (\mu + d\psi)$  is also a 1-step extension.*
- b) *If  $(f_1, f_2): \mathfrak{g} \rightarrow L_\infty(M, \omega)$  is a moment map and  $\psi: \mathfrak{g} \rightarrow C^\infty(M)$  is any linear map, then we obtain a new moment map with components*

$$\begin{aligned} \tilde{f}_1 &= f_1 + d\psi \\ \tilde{f}_2(x, y) &= f_2(x, y) + \psi([x, y]). \end{aligned}$$

*Proof.* A straightforward calculation left for the reader.  $\square$

**Remark 7.6.** In our work in progress [14], we also consider a notion of equivalence for moment maps whose equivalence classes are strictly larger than those arising in Lemma 7.5 b).

**Proposition 7.7.** *If  $G$  is compact and semisimple or, more generally,  $H^1(\mathfrak{g}, \mathbb{R}) = H^2(\mathfrak{g}, \mathbb{R}) = 0$ , then:*

- a) *Any two equivariant extensions are related as in Lemma 7.5 a,*
- b) *Any two moment maps are related as in Lemma 7.5 b.*

*Proof.* a) We have to show that given two 1-step extensions  $\omega - \mu, \omega' - \mu'$  there is a  $G$ -equivariant  $\tilde{\psi}: \mathfrak{g} \rightarrow C^\infty(M)$  such that  $\mu' = \mu + d\tilde{\psi}$ . Recall that by Thm. 6.3,  $f_1 = \mu$  and  $f_2(x, y) = \iota_{v_x}\mu(y)$  are the components of a moment map. Similarly, we obtain a moment map  $(f'_1, f'_2)$  using  $\mu'$ . Hence from Eq. (25), we see that  $(\mu' - \mu)([x, y])$  is an exact 1-form for all  $x, y \in \mathfrak{g}$ . From  $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ , which is equivalent to  $H^1(\mathfrak{g}, \mathbb{R}) = 0$ , we deduce that there exists a map  $\psi: \mathfrak{g} \rightarrow C^\infty(M)$  such that  $\mu' = \mu + d\psi$ .

The map  $\psi$  will not be equivariant in general. We now modify it suitably to obtain an equivariant map. Since  $\mu$  and  $\mu'$  are 1-step extensions of  $\omega$ , Eqs. (16) and (17) imply that we have for all  $x, y \in \mathfrak{g}$ :

$$(27) \quad \mathcal{L}_{v_x}d\psi(y) = d\psi([x, y]), \quad \iota_{v_x}d\psi(x) = \mathcal{L}_{v_x}\psi(x) = 0.$$

Define

$$c(x, y) = \mathcal{L}_{v_x}\psi(y) - \psi([x, y]).$$

The map  $c$  is clearly the obstruction to  $\psi$  being equivariant, and Eq. (27) implies that it is  $\mathbb{R}$ -valued and skew-symmetric.

We claim that  $c$  is a Lie algebra cocycle. We have

$$(d_{\mathfrak{g}}c)(x, y, z) = -(c([x, y], z) + c.p.) = (\mathcal{L}_{v_z}\psi([x, y]) - \psi([z, [x, y]])) + c.p.$$

This is zero by the Jacobi identity of  $\mathfrak{g}$  and because  $\mathfrak{g} \neq \mathfrak{f}$

$$\mathcal{L}_{v_z}\psi([x, y]) + c.p. = \iota_{v_z}(\mu' - \mu)([x, y]) + c.p. = (f_2(z, [x, y]) - f'_2(z, [x, y])) + c.p. = 0,$$

where the last equality uses Eq. (26).

From  $H^2(\mathfrak{g}, \mathbb{R}) = 0$  we know that there exists  $b: \mathfrak{g} \rightarrow \mathbb{R}$  such that  $c = d_{\mathfrak{g}}b$ , that is,  $c(x, y) = -b([x, y])$  for all  $x, y \in \mathfrak{g}$ . The map

$$\tilde{\psi} = \psi - b: \mathfrak{g} \rightarrow C^\infty(M)$$

clearly satisfies  $\mu' = \mu + d\tilde{\psi}$  (since it differs from  $\psi$  by a constant term), and it is equivariant since  $\mathcal{L}_{v_x}\tilde{\psi}(y) = \mathcal{L}_{v_x}\psi(y) = \tilde{\psi}([x, y])$ .

b) We have to show that given two moment maps with components  $(f_1, f_2)$  and  $(f'_1, f'_2)$  respectively, there is  $\psi: \mathfrak{g} \rightarrow C^\infty(M)$  such that  $f'_1 - f_1 = d\psi$  and  $(f'_2 - f_2)(x, y) = \psi([x, y])$ .

Notice that  $f'_2 - f_2: \Lambda^2\mathfrak{g} \rightarrow C^\infty(M)$  is a Lie algebra cocycle w.r.t. the *trivial* representation of  $\mathfrak{g}$  on  $C^\infty(M)$ , by Eq. (26). From  $H^2(\mathfrak{g}, C^\infty(M)) = H^2(\mathfrak{g}, \mathbb{R}) \otimes C^\infty(M) = 0$  we obtain a map  $\psi: \mathfrak{g} \rightarrow C^\infty(M)$  such that  $(f'_2 - f_2)(x, y) = \psi([x, y])$  for all  $x, y \in \mathfrak{g}$ . To conclude we just need to assure that the identity  $f'_1 - f_1 = d\psi$  holds: it does when both sides are applied to elements of the form  $[x, y] \in \mathfrak{g}$ , by Eq. (25) and the above. As  $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ , it holds for all elements of  $\mathfrak{g}$ .  $\square$

**7.5. Equivariant moment maps not arising from equivariant cocycles.** Let  $G$  act on the pre-2-plectic manifold  $(M, \omega)$ . Let  $\omega - \mu$  be an equivariant 1-step extension, and let  $(f_1, f_2): \mathfrak{g} \rightarrow L_\infty(M, \omega)$  be the corresponding equivariant moment map (Thm. 6.3). Here we explain how to modify this moment map to obtain a new *equivariant* moment map that does *not* arise from any equivariant cocycle.

If a linear map  $\tilde{f}_1: \mathfrak{g} \rightarrow \Omega^1(M)$

- takes values in closed 1-forms
- is equivariant:  $d(\iota_{v_x}\tilde{f}_1(y)) = \mathcal{L}_{v_x}\tilde{f}_1(y) = \tilde{f}_1([x, y])$  for all  $x, y \in \mathfrak{g}$ ,

then  $f_1 + \tilde{f}_1$  satisfies condition (A) at the beginning of Sec. 7.2 and is equivariant. If a skew-symmetric map  $\tilde{f}_2: \mathfrak{g} \otimes \mathfrak{g} \rightarrow C^\infty(M)$

- is equivariant
- satisfies  $\tilde{f}_1([x, y]) = d(\tilde{f}_2(x, y))$  for all  $x, y \in \mathfrak{g}$
- satisfies  $\tilde{f}_2(x, [y, z]) + c.p. = 0$  for all  $x, y, z \in \mathfrak{g}$ ,



then  $f_1 + \tilde{f}_1$  and  $f_2 + \tilde{f}_2$  are the components of a new equivariant moment map. (The last two conditions above guarantee Eq. (25) and (26) are satisfied). Furthermore, if we require that the constant function  $\iota_{v_x} \tilde{f}_1(x)$  is non-zero for some  $x \in \mathfrak{g}$ , then the last condition in Eq. (17) cannot be satisfied, and hence the new moment map can *not* arise from any equivariant cocycle. When  $\mathfrak{g}$  is an abelian Lie algebra and  $M$  is connected, the equivariance of  $\tilde{f}_1$  boils down to the condition that  $\iota_{v_x} \tilde{f}_1(y)$  is a constant function for all  $x, y \in \mathfrak{g}$ , and the three conditions on  $\tilde{f}_2$  simply imply that  $\tilde{f}_2$  takes values in the constant functions. Below we present a concrete instance of this construction:

**Example 7.8.** Let  $G$  be the abelian group  $S^1 \times S^1$ , and  $(M, \omega) = (S^1 \times S^1 \times \mathbb{R}, d\theta_1 \wedge d\theta_2 \wedge dz)$ . We take the infinitesimal action of  $\mathfrak{g}$  on  $M$  to be  $(1, 0) \in \mathfrak{g} \mapsto \partial_{\theta_1}$ ,  $(0, 1) \mapsto \partial_{\theta_2}$ . It is easily checked that  $\omega - \mu$  is a 1-step extension (see Eq. (16), Eq. (17)), where

$$\mu: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^1(M), \quad (1, 0) \mapsto zd\theta_2, (0, 1) \mapsto -zd\theta_1.$$

We can take  $\tilde{f}_1$  to be

$$\tilde{f}_1: \mathfrak{g} \rightarrow \Omega_{\text{closed}}^1(M), \quad (1, 0) \mapsto d\theta_1, (0, 1) \mapsto d\theta_2,$$

and any arbitrary skew-symmetric  $\tilde{f}_2: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{R}$ . Then, as seen above,  $f_1 + \tilde{f}_1$  and  $f_2 + \tilde{f}_2$  are the components of an equivariant moment map, which can not arise from any equivariant cocycle since  $\iota_{v_x} \tilde{f}_1(x) = 1 \neq 0$  for  $x = (1, 0)$ .

The discussion in this subsection proves:

**Proposition 7.9.** *Not all equivariant moment maps for actions on 2-plectic manifolds arise from 1-step extensions.*

**Remark 7.10.** In [14] we use Example 7.8 to obtain an equivariant moment map which is *not equivalent* (in the sense of [14], see Remark 7.6) to any moment map arising from a 1-step extension of  $\omega$ . This provides a statement stronger than Prop. 7.9 above.

## 8. EXAMPLES

In this section we present more examples of moment maps, many of which are generalizations of interesting examples from symplectic geometry. In light of Remark 6.4, all of them can be understood as arising from 1-step extensions as in Thm. 6.3, even if  $G$  is not compact.

In Sec. 10 we will give one more (infinite dimensional) example.

**8.1. Exact pre- $n$ -plectic forms.** Let  $M$  be a manifold with a  $G$ -action, let  $\alpha \in \Omega^n(M)^G$  be a  $G$ -invariant  $n$ -form and consider  $\omega = d\alpha$ .

**Lemma 8.1.** *If  $\mu \in (\Omega^{n-1}(M) \otimes \mathfrak{g}^*[-2])^G$  is defined by  $\mu(x) = \iota_{v_x} \alpha$ , then  $\omega - \mu$  is a 1- extension of  $\omega$ .*

*Proof.* It suffices to check that  $\mu(x)$  satisfies Eq. (16) and (17). Relation (16) comes from the Cartan relation  $\iota_{[v,w]} = [\mathcal{L}_v, \iota_w] = \mathcal{L}_v \circ \iota_w - \iota_w \circ \mathcal{L}_v$ , together with the fact that  $\mathcal{L}_{v_x} \alpha = 0$  for all  $x$  in  $\mathfrak{g}$ . Concerning (17), clearly  $d\omega = dd\alpha = 0$ . Moreover, the Cartan relation  $d \circ i_{v_x} + i_{v_x} \circ d = \mathcal{L}_{v_x}$ , together with the invariance of  $\alpha$  gives  $d\mu(x) = -i_{v_x} \omega$ . Finally the antisymmetry of  $\alpha$  gives  $i_{v_x} \mu(x) = 0$ .  $\square$

Hence by Thm 6.3 we obtain a homotopy moment map  $\mathfrak{g} \rightarrow L_\infty(M, \omega)$ , given by

$$f_k: \mathfrak{g}^{\otimes k} \rightarrow \Omega^{n-k}(M), \quad 1 \leq k \leq n$$

$$f_k(x_1, \dots, x_k) = (-1)^{k-1} \varsigma(k) \iota(v_{x_1} \wedge \dots \wedge v_{x_k}) \alpha$$

We present two concrete examples. The first one generalizes actions on cotangent bundles by cotangent lifts in symplectic geometry.

**Example 8.2** (Cotangent lifts). If  $G$  acts on a manifold  $N$  and  $n$  is an integer, take  $M = \Lambda^n T^*N$  and the  $G$ -action induced by the cotangent lift. Take  $\alpha \in \Omega^n(M)$  to be the canonical form defined by

$$\alpha(w_1, \dots, w_n)|_\xi = \xi(\pi_* w_1, \dots, \pi_* w_n) \quad \forall \xi \in M$$

where  $\pi: M \rightarrow N$  is the projection and  $w_1, \dots, w_n \in T_\xi M$ . The  $n$ -form  $\alpha$  is invariant, and it is known that  $d\alpha$  is an  $n$ -plectic form. (In the case  $n = 1$  it is, up to sign, the canonical symplectic form on  $T^*N$ ). From Lemma 8.1 we know that an equivariant extension  $\mu$  of  $\omega$  is given by  $\mu(x) = \iota_{v_x} \alpha$ , i.e.

$$\mu(x)(w_2, \dots, w_n)|_\xi = \xi((v_x)_{\pi(\xi)}, \pi_* w_2, \dots, \pi_* w_n),$$

where  $v_x$  denotes the fundamental vector field for the action on  $\Lambda^n T^*N$  (which restricts to the fundamental vector field for the action on  $N$ ).

In other words,  $(\mu(x))_\xi = \pi^*(\iota_{v_x} \xi)$ .

**Example 8.3** (Linear actions on vector spaces). It is well-known that any symplectic representation  $G \rightarrow \mathrm{GL}(V)$  on a symplectic vector space  $(V, \omega)$  is Hamiltonian. More precisely, if we denote the action of  $\xi \in \mathfrak{g}$  on  $V$  by  $\xi \cdot p$ , and consider the unique moment map  $J: V \rightarrow \mathfrak{g}^*$  vanishing at the origin, then its **components**  $J^\xi: V \rightarrow \mathbb{R} \quad \forall \xi \in \mathfrak{g}$  are the quadratic functions

$$J^\xi(p) = J(p)(\xi) = -\frac{1}{2}\omega(p, \xi \cdot p).$$

Below we generalize such actions to higher degree forms, and will see that the the analogs of the components for the moment map are no longer quadratic.

Let  $V$  be any vector space and  $\omega \in \Lambda^{n+1} V^*$ , giving rise to a constant pre- $n$ -plectic form on  $V$  (which is obviously closed and hence exact). Consider a linear action of a Lie group  $G \rightarrow \mathrm{GL}(V)$  preserving  $\omega$ . We claim that a  $G$ -invariant primitive of  $\omega$  is

$$\alpha = \frac{\iota_E \omega}{n+1}$$

where  $E$  is the Euler vector field on  $V$  (in coordinates,  $E = \sum_j x_k \frac{\partial}{\partial x_j}$ ). Indeed it is straightforward to check that  $d\iota_E \omega = \mathcal{L}_E \omega = (n+1)\omega$ . Also we have the equality  $\mathcal{L}_{v_\xi} \iota_E \omega = \iota_E \mathcal{L}_{v_\xi} \omega + \iota_{[v_\xi, E]} \omega = 0$  for all  $\xi \in \mathfrak{g}$ , since  $\omega$  is  $G$ -invariant and  $E$  commutes with all linear vector fields.

Hence, we have a moment map induced by the 1-step extension of  $\omega$  given in Lemma 8.1. Let us study this map in more detail. We denote by  $\phi: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  the Lie algebra morphism associated to the linear action. The infinitesimal generator of the action given by any  $\xi \in \mathfrak{g}$  is the linear vector field  $v_\xi|_p = -\phi(\xi)p$  (matrix multiplication). For all  $p \in V$  we have  $E|_p = p$ , so we obtain the following expression for the  $k$ -th component of moment map:

$$\begin{aligned} f_k(\xi_1, \dots, \xi_k)|_p &= (-1)^{k-1} \varsigma(k) \iota(v_{\xi_1} \wedge \dots \wedge v_{\xi_k}) \alpha|_p \\ &= -\varsigma(k) \frac{1}{n+1} \iota(p \wedge \phi(\xi_1)p \wedge \dots \wedge \phi(\xi_k)p) \omega. \end{aligned}$$

Notice that the coefficients of the  $(n-k)$ -form  $f_k(\xi_1, \dots, \xi_k)$  are polynomials of degree  $k+1$ .

The next example is a special case of Example 8.3 and generalizes the following simple case of Hamiltonian action on a symplectic manifold: the action of the circle on  $\mathbb{R}^2$  by rotations, with moment map  $(x_1, x_2) \mapsto -\frac{1}{2}(x_1^2 + x_2^2)$ .

**Example 8.4** ( $\mathrm{SO}(n)$ -action on  $\mathbb{R}^n$ ). We consider the canonical action of  $G = \mathrm{SO}(n)$  on  $\mathbb{R}^n$ , the latter endowed with the constant volume form

$$\omega = dx_1 \dots dx_n.$$

It is  $G$ -invariant, and by Example 8.3 an invariant primitive is

$$\alpha = \frac{1}{n} \sum_{k=1}^n (-1)^{k+1} x_k dx_1 \dots \widehat{dx_k} \dots dx_n.$$

A basis of the Lie algebra  $\mathfrak{so}(n)$  is  $\{e_{ij} : 1 \leq i < j \leq n\}$ , where  $e_{ij}$  denotes the matrix with  $-1$  in the  $(i, j)$ -th position,  $1$  in the  $(j, i)$ -th position and zeros elsewhere. The corresponding generators of the action are given by

$$v_{ij} = x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j}.$$

By Lemma 8.1 we know that an equivariant 1-step extension  $\mu$  of  $\omega$  is given by

$$\begin{aligned} \mu(e_{ij}) &= \iota_{v_{ij}} \alpha \\ &= \frac{1}{n} \left( \sum_{k=1}^{i-1} - \sum_{k=i+1}^n \right) (-1)^{k+1+i} x_j x_k dx_1 \dots \widehat{dx_i} \dots \widehat{dx_k} \dots dx_n \\ &\quad - \frac{1}{n} \left( \sum_{k=1}^{j-1} - \sum_{k=j+1}^n \right) (-1)^{k+1+j} x_i x_k dx_1 \dots \widehat{dx_j} \dots \widehat{dx_k} \dots dx_n, \end{aligned}$$

and that we can build a moment map out of  $\mu$ . Notice that, in the symplectic case ( $n = 2$ ), one recovers  $\mu(e_{12}) = -\frac{1}{2}(x_1^2 + x_2^2)$ .

**8.2.  $\mathrm{SO}(3)$  action on the 3-sphere.** A classical example of a Hamiltonian action in symplectic geometry is the action of  $S^1 = \mathrm{SO}(2)$  on  $S^2$  by rotations about the  $z$ -axis. The “height function”  $-z : S^2 \rightarrow \mathbb{R}$  provides a moment map.

We now extend this to  $n = 3$ . As we wish to extend this to arbitrary values of  $n$  in the future, we carry out part of our computations for arbitrary  $n$ . The unit sphere  $M = S^n \subset \mathbb{R}^{n+1}$  is endowed with the volume form  $\omega$  obtained restricting

$$\sum_{k=1}^{n+1} (-1)^{k+1} x_k dx_1 \dots \widehat{dx_k} \dots dx_{n+1} = \alpha \wedge dx_{n+1} + \frac{(-1)^n}{n} x_{n+1} \cdot d\alpha,$$

where

$$\alpha = \sum_{k=1}^n (-1)^{k+1} x_k dx_1 \dots \widehat{dx_k} \dots dx_n.$$

View  $G = \mathrm{SO}(n)$  as a subgroup of  $\mathrm{SO}(n+1)$  (embedded as matrices with a “1” in the lower right corner), and consider the obvious action on  $S^n \subset \mathbb{R}^{n+1}$  by matrix multiplication.

This action preserves  $\omega$ , for in Ex. 8.4 we saw that it preserves  $\alpha$ . With the notations of Ex. 8.4, we have

$$\iota_{v_{ij}} \omega = d \left( \frac{(-1)^{n+1}}{n} \iota_{v_{ij}} \alpha \cdot x_{n+1} \right) + \frac{n+1}{n} \iota_{v_{ij}} \alpha \wedge dx_{n+1},$$

using the fact that the action preserves  $\alpha$ .

To write out the right-most term we consider  $\iota_{v_{ij}} \alpha$ . Using the fact that the function  $\sum_{k=1}^{n+1} x_k^2$  equals one on  $S^n$ , and therefore the pullback to  $S^n$  of its differential  $2 \sum_{k=1}^{n+1} x_k dx_k$  vanishes, a lengthy computation shows that

$$\iota_{v_{ij}} \alpha = (-1)^{i+j} (1 - x_{n+1}^2) dx_1 \dots \widehat{dx_i} \dots \widehat{dx_j} \dots dx_n + (\text{terms containing } dx_{n+1}).$$

Hence

$$\iota_{v_{ij}} \alpha \wedge dx_{n+1} = (-1)^{i+j+n} d \left( (x_{n+1} - \frac{1}{3} x_{n+1}^3) dx_1 \dots \widehat{dx_i} \dots \widehat{dx_j} \dots dx_n \right).$$

A primitive for  $-\iota_{v_{ij}}\omega$  is therefore

$$(28) \quad \mu(e_{ij}) = \frac{(-1)^n}{n} x_{n+1} \cdot \iota_{v_{ij}}\alpha + (-1)^{i+j+n+1} \frac{n+1}{n} \left( x_{n+1} - \frac{1}{3} x_{n+1}^3 \right) dx_1 \dots \widehat{dx_i} \dots \widehat{dx_j} \dots dx_n.$$

We check when  $\mu \in \Omega^{n-1}(M) \otimes \mathfrak{g}^*$  provides a 1-step extension of  $\omega$  (see Eq. (16) and (17)). The condition

$$\mathcal{L}_{v_{ij}}\mu(e_{i'j'}) = \mu([e_{ij}, e_{i'j'}]),$$

for all  $i < j$  and  $i' < j'$ , follows from a computation that uses the identity  $[\mathcal{L}_v, \iota_w] = \iota_{[v,w]}$  and a careful care of signs. The condition  $\iota_{v_x}\mu(x) = 0$  for all  $x \in \mathfrak{g}$  is equivalent to  $\iota_{v_x}\mu(y)$  being skew-symmetric in  $x$  and  $y$  for all  $x, y \in \mathfrak{g}$ . It is sufficient to check the skew-symmetry on pairs of elements of the basis  $\{e_{ij} : 1 \leq i < j \leq n\}$  of  $\mathfrak{g}$ . For  $n = 3$  one computes easily that the skew-symmetry condition is satisfied. Hence, by Thm. 6.3, for  $n = 3$  from  $\mu$  we obtain a homotopy moment map.

**Remark 8.5.** When  $n \geq 4$  we have  $\iota_{v_{12}}\mu(e_{34}) \neq -\iota_{v_{34}}\mu(e_{12})$ , hence  $\omega - \mu$  is not a 1-step extension.

**Remark 8.6.** Writing out explicitly Eq. (28) we obtain

$$\begin{aligned} \mu(e_{ij}) &= (-1)^{i+j+n+1} \left( x_{n+1} - \frac{n-2}{3n} x_{n+1}^3 \right) dx_1 \dots \widehat{dx_i} \dots \widehat{dx_j} \dots dx_n \\ &\quad + \frac{(-1)^{i+j+1}}{n} \left( \sum_{k=1}^{i-1} - \sum_{k=i+1}^{j-1} + \sum_{k=j+1}^n \right) (-1)^{k-1} x_k x_{n+1}^2 \cdot dx_1 \dots \widehat{dx_i} \dots \widehat{dx_j} \dots \widehat{dx_k} \dots dx_{n+1}. \end{aligned}$$

Notice that the cubic terms disappear only in the case  $n = 2$ .

**8.3. Adjoint action and conjugacy classes.** Let  $G$  be a compact Lie group whose Lie algebra  $\mathfrak{g}$  is equipped with an Ad-invariant inner-product  $\langle \cdot, \cdot \rangle$ . Then  $G$  equipped with the bi-invariant Cartan 3-form

$$\omega = \frac{1}{12} \langle \theta_L, [\theta_L, \theta_L] \rangle = \frac{1}{12} \langle \theta_R, [\theta_R, \theta_R] \rangle$$

is a pre-2-plectic manifold. Here  $\theta_L$  and  $\theta_R$  are, respectively, the left and right invariant Maurer-Cartan 1-forms on  $G$  satisfying

$$d\theta_L + \frac{1}{2}[\theta_L, \theta_L] = 0 \quad d\theta_R - \frac{1}{2}[\theta_R, \theta_R] = 0.$$

If  $G$  is, for example, also semi-simple, then  $\omega$  is non-degenerate and hence  $(G, \omega)$  is 2-plectic.

Clearly, the action of  $G$  on itself via conjugation preserves  $\omega$ , and this action gives rise to a homotopy moment map. The Hamiltonian vector field associated to  $x \in \mathfrak{g}$  is

$$v_x = v_x^L - v_x^R,$$

where  $v^L$  and  $v^R$  are, respectively, the left and right invariant vector fields on  $G$  associated to  $x$ . A straightforward calculation using the above Maurer-Cartan equations and the identity  $\text{Ad}_g \theta_L = \theta_R$  gives:

$$\frac{1}{2} d\langle \theta_L + \theta_R, x \rangle = -\iota(v_x)\omega.$$

The fact that this action lifts to a moment map follows from Theorem 6.3 and the well-known fact that the  $\mathfrak{g}^*$ -valued 1-form

$$\mu(x) = \frac{1}{2} \langle \theta_L + \theta_R, x \rangle \quad \forall x \in \mathfrak{g}$$

gives an equivariant extension  $\omega - \mu$  of the Cartan 3-form.

Let us write out the structure map  $f_2: \mathfrak{g} \otimes \mathfrak{g} \rightarrow C^\infty(G)$  explicitly for this case. By definition, at a point  $g \in G$ , we have

$$\begin{aligned} f_2(x, y)(g) &= \iota(v_x)\mu(y)|_g = \frac{1}{2}\langle \theta_L(v_x) + \theta_R(v_x), y \rangle|_g \\ &= \frac{1}{2}\langle (\text{Ad}_g - \text{Ad}_{g^{-1}})x, y \rangle. \end{aligned}$$

This piece of the moment map is related to an interesting invariant 2-form defined on the conjugacy classes of  $G$ . The Hamiltonian vector fields  $v_x$  are minus the fundamental vector fields associated to the conjugation action, and therefore span the tangent spaces of the conjugacy classes. It follows from Proposition 3.1 in [1] that if  $\iota_C: C \hookrightarrow G$  is the inclusion of a conjugacy class, then

$$dB = -\iota_C^* \omega$$

where  $B \in \Omega^2(C)^G$  is

$$B_g(v_x, v_y) = f_2(x, y)(g) \quad \forall g \in C \quad \forall x, y \in \mathfrak{g}.$$

Conjugacy classes are important examples of “quasi-Hamiltonian  $G$ -spaces” [1], just as coadjoint orbits are examples of Hamiltonian  $G$ -spaces in symplectic geometry. As the above example suggests, it may be interesting to investigate further the relationship between quasi-Hamiltonian  $G$ -spaces and homotopy moment maps.

## 9. OBSTRUCTIONS AND CENTRAL EXTENSIONS

Here we describe an obstruction to the existence of moment maps characterized by a class in Lie algebra cohomology (Thm. 9.6). Conversely, we show that if both the class and certain de Rham cohomology groups vanish, then a moment map always exists (Thm. 9.7). If the obstruction does not vanish, we obtain a  $L_\infty$ -morphism into  $\text{PoissonLie}(M, \omega)$  not from the Lie algebra, but from a Lie  $n$ -algebra which can be described a ‘higher central extension’ (Prop. 9.11).

Throughout this section, we assume we have a Lie group  $G$  acting on a pre- $n$ -plectic manifold  $(M, \omega)$  such that  $\omega$  is preserved via infinitesimal diffeomorphisms i.e. the Lie algebra  $\mathfrak{g}$  acts via local Hamiltonian vector fields:

$$\mathcal{L}_{v_x} \omega = 0,$$

giving us the usual Lie algebra morphism

$$\mathfrak{g} \rightarrow \mathfrak{X}_{\text{LHam}}(M), \quad x \mapsto v_x.$$

**9.1. Lie algebra cohomology.** For any Lie algebra  $\mathfrak{k}$ , the Chevalley-Eilenberg differential on the cochain complex  $\text{CE}(\mathfrak{k}) = \text{Hom}(\Lambda^\bullet \mathfrak{k}, \mathbb{R})$  is:

$$\delta_{\text{CE}}(c)(x_1, \dots, x_{n+1}) = \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} c([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{n+1}).$$

**Remark 9.1.** If  $(f_k): \mathfrak{g} \rightarrow L_\infty(M, \omega)$  is a moment map, then the structure maps  $f_k$  must satisfy Eqs. (14) and (15). Observe that the left-hand side of both of these equations can be written as

$$-(\delta_{\text{CE}} f_{k-1})(x_1, \dots, x_k).$$

The relationship between Chevalley-Eilenberg cohomology and homotopy moment maps is the starting point of the work in [14].

Before considering  $G$ -actions, we make an observation about the Lie algebra of local Hamiltonian vector fields. The following proposition says that  $\omega$  determines a class in  $H_{\text{CE}}^{n+1}(\mathfrak{X}_{\text{LHam}}(M))$ .

**Proposition 9.2.** *If  $(M, \omega)$  is a pre- $n$ -plectic manifold, then  $\forall p \in M$  the linear map*

$$c_p: \Lambda^{n+1} \mathfrak{X}_{\text{LHam}}(M) \rightarrow \mathbb{R}$$

$$v_1 \wedge \cdots \wedge v_{n+1} \mapsto (-1)^n \zeta(n+1) \iota(v_1 \wedge \cdots \wedge v_{n+1}) \omega|_p,$$

*is a degree  $(n+1)$ -cocycle in  $\text{CE}(\mathfrak{X}_{\text{LHam}}(M))$ . Moreover, if  $M$  is connected, then the cohomology class  $[c_p]$  is independent of  $p \in M$ .*

To prove the above proposition, we need the following technical lemma which generalizes [25, Lem. 3.7][34, Lem. 6.8].

**Lemma 9.3.** *If  $(M, \omega)$  is a pre- $n$ -plectic manifold and  $v_1, \dots, v_m \in \mathfrak{X}_{\text{LHam}}(M)$  with  $m \geq 2$  then*

$$(29) \quad d\iota(v_1 \wedge \cdots \wedge v_m) \omega =$$

$$(-1)^m \sum_{1 \leq i < j \leq m} (-1)^{i+j} \iota([v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_m) \omega.$$

*Proof.* We proceed via induction on  $m$ . For  $m = 2$ , Prop. 4.5 implies Eq. (29) holds.

Assume Eq. (29) holds for  $m - 1$ . Since  $\iota(v_1 \wedge \cdots \wedge v_m) = \iota_{v_m} \iota(v_1 \wedge \cdots \wedge v_{m-1})$ , Eq. (4) implies:

$$(30) \quad d\iota(v_1 \wedge \cdots \wedge v_m) \omega = \mathcal{L}_{v_m} \iota(v_1 \wedge \cdots \wedge v_{m-1}) \omega - \iota_{v_m} d\iota(v_1 \wedge \cdots \wedge v_{m-1}) \omega.$$

We can rewrite the first term on the right hand side as

$$\begin{aligned} \mathcal{L}_{v_m} \iota(v_1 \wedge \cdots \wedge v_{m-1}) \omega &= \iota([v_m, v_1 \wedge \cdots \wedge v_{m-1}]) \omega + \iota(v_1 \wedge \cdots \wedge v_{m-1}) \mathcal{L}_{v_m} \omega \\ &= \iota([v_m, v_1 \wedge \cdots \wedge v_{m-1}]) \omega \\ &= \sum_{i=1}^{m-1} (-1)^i \iota([v_i, v_m] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{m-1}) \omega. \end{aligned}$$

Here in the first equality we used Eq. (5), in the second the fact that  $v_m$  is locally Hamiltonian, and in the third the definition of the Schouten bracket (see Eq. (2)) in the form

$$[v_m, v_1 \wedge \cdots \wedge v_{m-1}] = \sum_{i=1}^{m-1} (-1)^{i+1} [v_m, v_i] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{m-1}.$$

Combining this with the second term in Eq. (30) and using the inductive hypothesis gives

$$\begin{aligned} &d\iota(v_1 \wedge \cdots \wedge v_m) \omega \\ &= \sum_{i=1}^{m-1} (-1)^i \iota([v_i, v_m] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{m-1}) \omega \\ &\quad - (-1)^{m-1} \sum_{1 \leq i < j \leq m-1} (-1)^{i+j} \iota_{v_m} \iota([v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_{m-1}) \omega \\ &= (-1)^m \left( \sum_{i=1}^{m-1} (-1)^{i+m} \iota([v_i, v_m] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{m-1}) \omega \right. \\ &\quad \left. + \sum_{1 \leq i < j \leq m-1} (-1)^{i+j} \iota([v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_m) \omega \right) \\ &= (-1)^m \sum_{1 \leq i < j \leq m} (-1)^{i+j} \iota([v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_m) \omega. \end{aligned}$$

□

Now we have what we need to prove Prop. 9.2.



*Proof of Prop. 9.2.* Clearly  $c_p \in \text{Hom}(\Lambda^{n+1} \mathfrak{X}_{\text{LHam}}(M), \mathbb{R})$ . We compute:

$$(31) \quad \delta_{\text{CE}}(c_p)(x_1, \dots, x_{n+2}) = \varsigma(n+1) \sum_{1 \leq i < j \leq n+2} (-1)^{n+i+j} \iota([v_i, v_j] \wedge v_1 \wedge \dots \wedge \hat{v}_i \wedge \dots \wedge \hat{v}_j \wedge \dots \wedge v_{n+2}) \omega|_p.$$

We use Lemma 9.3 for  $m = n + 2$ . The right-hand side of of Eq. (31) above is equal to plus or minus the right-hand side of Eq. 29 evaluated at the point  $p$ . However, the left-hand side of Eq. (29) vanishes because  $\omega \in \Omega^{n+1}(M)$ . Hence,  $\delta_{\text{CE}}(c_p) = 0$ .

Now, assume  $M$  is connected, and let  $p' \in M$ . There exists a path  $\gamma: [0, 1] \rightarrow M$  such that  $\gamma(0) = p$  and  $\gamma(1) = p'$ . We define a map  $b: \Lambda^n \mathfrak{X}_{\text{LHam}}(M) \rightarrow \mathbb{R}$  by

$$b(v_1, \dots, v_n) = -\varsigma(n+1) \int_{\gamma} \iota(v_1 \wedge \dots \wedge v_n) \omega.$$

It follows from Lemma 9.3 that

$$d\iota(v_1 \wedge \dots \wedge v_{n+1}) \omega = (-1)^{n+1} \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \iota([v_i, v_j] \wedge v_1 \wedge \dots \wedge \hat{v}_i \wedge \dots \wedge \hat{v}_j \wedge \dots \wedge v_{n+1}) \omega.$$

Integrating both sides of the above equation over  $\gamma$  gives

$$\iota(v_1 \wedge \dots \wedge v_{n+1}) \omega|_{p'} - \iota(v_1 \wedge \dots \wedge v_{n+1}) \omega|_p = (-1)^n \varsigma(n+1) \delta_{\text{CE}}(b)(v_1, \dots, v_{n+1}),$$

and, hence,  $c_{p'} - c_p = \delta_{\text{CE}} b$ .  $\square$

If  $G$  is acting on  $(M, \omega)$ , then Prop. 9.2 gives an important corollary.

**Corollary 9.4.** *If  $(M, \omega)$  is a pre- $n$ -plectic manifold equipped with a  $G$  action such that  $\mathfrak{g}$  preserves  $\omega$  then  $\forall p \in M$  the linear map*

$$c_p^{\mathfrak{g}}: \Lambda^{n+1} \mathfrak{g} \rightarrow \mathbb{R} \\ x_1 \wedge \dots \wedge x_{n+1} \mapsto (-1)^n \varsigma(n+1) \iota(v_1 \wedge \dots \wedge v_{n+1}) \omega|_p,$$

where  $v_i$  is the vector field associated to  $x_i \in \mathfrak{g}$ , is a degree  $(n+1)$ -cocycle in  $\text{CE}(\mathfrak{g})$ . Moreover, if  $M$  is connected, then the cohomology class  $[c_p^{\mathfrak{g}}]$  is independent of  $p \in M$ .

*Proof.* By assumption,  $\mathfrak{g}$  acts via local Hamiltonian vector fields, and  $c_p^{\mathfrak{g}}$  is the pullback of the cocycle defined in Prop. 9.2 along the Lie algebra morphism  $v_-$ .  $\square$

**Remark 9.5.** Note that if the  $G$ -action has a fixed point then  $[c_p^{\mathfrak{g}}] = 0$ .

The next proposition shows that the class  $[c_p^{\mathfrak{g}}] \in H_{\text{CE}}^{n+1}(\mathfrak{g})$  is an obstruction to having a homotopy moment map.

**Proposition 9.6.** *If  $(M, \omega)$  is a connected pre- $n$ -plectic manifold, and  $M$  is equipped with a  $G$  action which induces a homotopy moment map  $\mathfrak{g} \rightarrow L_{\infty}(M, \omega)$ , then*

$$[c_p^{\mathfrak{g}}] = 0$$

where  $[c_p^{\mathfrak{g}}] \in H_{\text{CE}}^{n+1}(\mathfrak{g})$  is the cohomology class defined in Cor. 9.4.

*Proof.* By Def. 5.1 the homotopy moment map corresponds to structure maps  $f_1, \dots, f_n$  satisfying Eqs. (14) and (15). Since  $|f_n| = 1 - n$ , the map  $f_n$  takes values in  $C^{\infty}(M)$ . Let  $p \in M$ , and define

$$b(x_1, \dots, x_n) = (-1)^{n+1} f_n(x_1, \dots, x_n)|_p.$$

Clearly,  $b \in \text{Hom}(\Lambda^n \mathfrak{g}, \mathbb{R})$ . Eq. (15). then implies

$$(c_p^{\mathfrak{g}})(x_1, \dots, x_{n+1}) = \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} b([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{n+1}).$$

Hence,  $c_p^{\mathfrak{g}} = \delta_{\text{CE}} b$ . □

**9.2. Lifting  $\mathfrak{g}$ -actions to moment maps.** Recall from Prop. 4.8 that there is a surjective (and strict)  $L_\infty$ -morphism

$$\pi: \text{PoissonLie}(M, \omega) \twoheadrightarrow \mathfrak{X}_{\text{Ham}}(M)$$

which is simply the projection  $(v, \alpha) \mapsto v$  in degree 0. Suppose we have a Lie group  $G$  acting on  $(M, \omega)$ , such that the infinitesimal action of  $\mathfrak{g}$  is via Hamiltonian vector fields. Exhibiting a moment map for such an action means finding a lift

$$(32) \quad \begin{array}{ccc} & \text{PoissonLie}(M, \omega) & \\ & \downarrow \pi & \\ \mathfrak{g} & \xrightarrow{v_-} & \mathfrak{X}_{\text{Ham}}(M) \end{array}$$

in the category of  $L_\infty$ -algebras. Since  $\mathfrak{g}$  acts by Hamiltonian vector fields there always exists a (non-unique) degree zero *linear map*

$$\begin{aligned} \mathfrak{g} &\rightarrow \text{PoissonLie}(M, \omega) \\ x &\mapsto (v_x, \phi(x)) \in \mathfrak{X}_{\text{Ham}}(M) \oplus \Omega_{\text{Ham}}^{n-1}(M) \end{aligned}$$

such that  $d\phi(x) = -\iota_{v_x}\omega$ . When does such a linear map lift to an  $L_\infty$ -morphism? Theorem 9.6 implies that it is necessary that the cohomology class  $[c_p^{\mathfrak{g}}]$  vanish. The next theorem shows that when certain topological assumptions are satisfied, this is also sufficient.

**Theorem 9.7.** *Let  $(M, \omega)$  be a connected pre- $n$ -plectic manifold equipped with a  $G$  action such that  $\mathfrak{g}$  acts via Hamiltonian vector fields. Let*

$$\phi: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^{n-1}(M)$$

*be any linear map such that  $d\phi(x) = -\iota_{v_x}\omega$  for all  $x \in \mathfrak{g}$ . If  $H_{\text{dR}}^i(M) = 0$  for  $1 \leq i \leq n-1$  and  $[c_p^{\mathfrak{g}}] = 0$ , where  $[c_p^{\mathfrak{g}}] \in H_{\text{CE}}^{n+1}(\mathfrak{g})$  is the cohomology class defined in Cor. 9.4, then there exists a homotopy moment map*

$$(f_k): \mathfrak{g} \rightarrow L_\infty(M, \omega)$$

*such that*

$$f_1 = \phi.$$

*Proof.* Let  $f_1 = \phi$ , implying that  $d(f_1(x)) = -\iota_{v_x}\omega$  for all  $x \in \mathfrak{g}$ . Notice that this equation is what is obtained allowing  $k = 1$  in Eq. (14) (taking  $f_{-1} = 0$ ). We now find recursively solutions for the equations appearing in (14).

**Claim 1:** *For every  $2 \leq k \leq n+1$ , if  $f_{k-1}$  satisfies Eq. (14) for  $k-1$ , then*

$$(33) \quad \sum_{1 \leq i < j \leq k} (-1)^{i+j+1} f_{k-1}([x_i, x_j], x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_k) - \varsigma(k) \iota(v_1 \wedge \dots \wedge v_k) \omega$$

*is a closed  $n+1-k$ -form for all  $x_1, \dots, x_k \in \mathfrak{g}$ .*

To prove the claim we proceed as follows. We have

$$\begin{aligned}
& d\iota(v_1 \wedge \cdots \wedge v_k)\omega \\
&= (-1)^k \sum_{1 \leq i < j \leq k} (-1)^{i+j} \iota([v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k)\omega \\
&= (-1)^k \sum_{1 \leq i < j \leq k} (-1)^{i+j} \varsigma(k-1) \left( -(\delta_{\text{CE}} f_{k-2})([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_k) \right. \\
&\quad \left. - df_{k-1}([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_k) \right) \\
&= \underbrace{(-1)^k \varsigma(k-1)}_{=\varsigma(k)} \left( (-\delta_{\text{CE}}^2 f_{k-2})(x_1, \dots, x_k) \right. \\
&\quad \left. + d \sum_{1 \leq i < j \leq k} (-1)^{i+j+1} f_{k-1}([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_k) \right)
\end{aligned}$$

using Lemma 9.3 in the first equality, and in the second the fact that  $f_{k-1}$  satisfies Eq. (14) for  $k-1$  as well as Remark 9.1. Since the Chevalley-Eilenberg differential  $\delta_{\text{CE}}$  squares to zero, the claim follows.

**Claim 2:** For all  $2 \leq k \leq n$ , there exist  $f_k: \Lambda^k \mathfrak{g} \rightarrow \Omega^{n-k}(M)$  satisfying Eq. (14) for  $k$ .

We prove Claim 2 by induction on  $k$ . The case  $k=1$  holds, as seen earlier, with  $f_1 = \phi$ . We fix  $2 \leq k \leq n$ . By the induction assumption we are allowed to apply Claim 1 for  $k$ . The assumption  $H^{n+1-k}(M) = 0$  implies that there exists  $f_k: \Lambda^k \mathfrak{g} \rightarrow \Omega^{n-k}(M)$  such that  $f_k(x_1, \dots, x_k)$  is a primitive for the  $n+1-k$ -form (33), for all  $x_1, \dots, x_k \in \mathfrak{g}$ . Equivalently,  $f_k$  satisfies Eq. (14) for  $k$ , proving Claim 2.

In general,  $f_n$  will not satisfy Eq. (15). It will iff  $h \in \text{Hom}(\Lambda^{n+1} \mathfrak{g}, C^\infty(M))$  vanishes, where

$$h(x_1, \dots, x_{n+1}) = (\delta_{\text{CE}} f_n)(x_1, \dots, x_{n+1}) + \varsigma(n+1) \iota(v_1 \wedge \cdots \wedge v_{n+1})\omega.$$

(Here we used Remark 9.1.) Now fix  $p \in M$ . We evaluate both summands of  $h$  at  $p$ , and obtain two elements of  $\text{Hom}(\Lambda^{n+1} \mathfrak{g}, \mathbb{R})$ : the first one is  $\delta_{\text{CE}}$ -exact by construction, the second is equal to  $\pm c_p^{\mathfrak{g}}$ , hence it is  $\delta_{\text{CE}}$ -exact by assumption. This means that there exists  $b \in \text{Hom}(\Lambda^n \mathfrak{g}, \mathbb{R})$  such that  $h|_p = \delta_{\text{CE}} b$ . However by Claim 1 (for  $k=n+1$ ) we know that  $h(x_1, \dots, x_{n+1})$  is a closed zero form for all  $x_1, \dots, x_{n+1} \in \mathfrak{g}$ , and since  $M$  is connected this means that  $h$  lies in  $\text{Hom}(\Lambda^{n+1} \mathfrak{g}, \mathbb{R})$ . Hence

$$h = \delta_{\text{CE}} b \in \text{Hom}(\Lambda^{n+1} \mathfrak{g}, \mathbb{R}).$$

Replacing  $f_n$  by  $f_n - b$  we therefore obtain a solution of Eq. (15), which still satisfies Eq. (14) for  $n$  as  $b$  takes values in the constants. We conclude that  $f_1, \dots, f_{n-1}, f_n - b$  are the components of a homotopy moment map.  $\square$

**Remark 9.8.** Note that the above proof does not require that the equality of cohomology classes  $[c_p^{\mathfrak{g}}] = [c_{p'}^{\mathfrak{g}}]$  hold for  $p, p' \in M$ . To have the theorem, one only needs to find  $p \in M$  such that  $[c_p^{\mathfrak{g}}] = 0$  and verify that the assumptions on de Rham cohomology and connectedness hold. Also, note that  $G$  need not be finite-dimensional here; the theorem also applies to actions by locally exponential infinite-dimensional Lie groups. In Sec. 10, we consider a case in which  $G$  is such a group acting on a pre- $n$ -plectic locally convex topological vector space.

**Remark 9.9.** The assumptions of Thm. 9.7 can be weakened; in fact, only particular components of  $H_{\text{CE}}^\bullet(\mathfrak{g}) \otimes H_{\text{dR}}^\bullet(M)$  need to vanish [14].

**9.3. Central  $n$ -extensions.** If  $(M, \omega)$  is a connected symplectic manifold, then Kostant's construction [19] gives a morphism of central extensions

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ \widehat{\mathfrak{g}} & \longrightarrow & C^\infty(M) \\ \downarrow & & \downarrow \pi \\ \mathfrak{g} & \xrightarrow{v_-} & \mathfrak{X}_{\text{Ham}}(M) \end{array}$$

where  $\widehat{\mathfrak{g}}$  is the central extension corresponding to the 2-cocycle  $c_p^{\mathfrak{g}}$ . This central extension is non-trivial iff there is no moment map which lifts the  $\mathfrak{g}$  action.

Now we describe how these ideas generalize to higher cases. First we recall a theorem of Baez and Crans [3, Thm. 55]: Given a Lie algebra  $(\mathfrak{g}, [\cdot, \cdot])$  and a degree  $(n+1)$ -cocycle  $c: \Lambda^{n+1}\mathfrak{g} \rightarrow \mathbb{R}$ , there exists a Lie  $n$ -algebra whose underlying complex is  $\mathfrak{g}$  in degree 0,  $\mathbb{R}$  in degree  $1-n$  and 0 in all other degrees. The structure maps are trivial except in degree zero where we have:

$$\begin{aligned} l_2(x_1, x_2) &= [x_1, x_2] \\ l_{n+1}(x_1, \dots, x_{n+1}) &= c(x_1, \dots, x_{n+1}) \\ l_k &= 0 \quad \text{if } k \neq 2, k \neq n+1. \end{aligned}$$

We call this Lie  $n$ -algebra a **central  $n$ -extension** of  $\mathfrak{g}$  and denote it by  $\widehat{\mathfrak{g}}_c$ . If  $c$  and  $c'$  are two such cocycles which differ by a coboundary, then the corresponding Lie  $n$ -algebras are quasi-isomorphic (Cor. A.9). If  $n = 1$ , then we recover the usual notion of central extension by setting  $l_2 = [\cdot, \cdot] + c$ . Let  $\pi_{\mathfrak{g}}: \widehat{\mathfrak{g}}_c \rightarrow \mathfrak{g}$  denote the projection. It clearly lifts to a strict  $L_\infty$ -morphism.

**Proposition 9.10.** *The short exact sequence of complexes*

$$\mathbb{R}[n-1] \rightarrow \widehat{\mathfrak{g}}_c \xrightarrow{\pi_{\mathfrak{g}}} \mathfrak{g}$$

*lifts to a strict exact sequence in the category of  $L_\infty$ -algebras.*

The following proposition is the higher analog of Kostant's construction in symplectic geometry for central extensions such as the Heisenberg Lie algebra.

**Proposition 9.11.** *Let  $(M, \omega)$  be a connected pre- $n$ -plectic manifold equipped with a  $G$  action such that  $\mathfrak{g}$  acts via Hamiltonian vector fields and let  $p \in M$ . Assume  $H_{\text{dR}}^k(M) = 0$  for  $1 \leq k \leq n-1$ . If  $\widehat{\mathfrak{g}}$  is the central  $n$ -extension constructed from the  $(n+1)$ -cocycle  $c_p^{\mathfrak{g}}$  defined in Cor. 9.4, then there exists an  $L_\infty$ -morphism*

$$(f_i): \widehat{\mathfrak{g}} \rightarrow \text{PoissonLie}(M, \omega)$$

*such that the following diagram (strictly) commutes*

$$(34) \quad \begin{array}{ccc} \widehat{\mathfrak{g}} & \xrightarrow{(f_i)} & \text{PoissonLie}(M, \omega) \\ \pi_{\mathfrak{g}} \downarrow & & \downarrow \pi \\ \mathfrak{g} & \xrightarrow{v_-} & \mathfrak{X}_{\text{Ham}}(M) \end{array}$$

*Proof.* We shall produce maps  $f_1, \dots, f_n$  such that equalities given Prop. A.8 are satisfied. Since  $\mathfrak{g}$  acts by Hamiltonian vector fields, there exists a linear map  $\phi: \mathfrak{g} \rightarrow \widehat{\Omega}_{\text{Ham}}^{n-1}(M)$  with  $\phi(x) = (v_x, \alpha_x)$  such that  $d\alpha_x = -\iota_{v_x}\omega$ . The map

$$\begin{aligned} f_1(x) &= \phi(x) \quad \forall x \in \mathfrak{g} \\ f_1(r) &= (-1)^n r \in C^\infty(M) \quad \forall r \in \mathbb{R} \end{aligned}$$

gives a degree 0 chain map  $f_1$  from the underlying complex of  $\widehat{\mathfrak{g}}$  to that of  $\text{PoissonLie}(M, \omega)$ .

We then proceed as we did in the first part of the proof of Thm. 9.7. Namely, since all closed  $k$ -forms have a primitive for  $0 \leq k \leq n-1$ , we inductively obtain maps  $f_i: \Lambda^i \mathfrak{g} \rightarrow \Omega^{n-i}(M)$  for  $i = 2, \dots, n$  such that Eq. (70) is satisfied. Let  $b: \Lambda^n \mathfrak{g} \rightarrow \mathbb{R}$  be

$$b(x_1, \dots, x_n) = f_n(x_1, \dots, x_n)|_p,$$

and let  $\tilde{f}_n = f_n - b$ .

Since  $db(x_1, \dots, x_n) = 0$  for all  $x_i$ , the map  $\tilde{f}_n$  also satisfies (70). It remains to show that Eq. (71) holds i.e. given  $x_1, \dots, x_{n+1} \in \mathfrak{g}$ , the function

$$\begin{aligned} C = \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+1} \tilde{f}_n([x_i, x_j], x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_{n+1}) \\ + (-1)^n c_p^{\mathfrak{g}}(x_1, \dots, x_{n+1}) - \varsigma(n+1) \iota(v_1 \wedge \dots \wedge v_{n+1}) \omega \end{aligned}$$

vanishes. Using Lemma 9.3 and Eq. (70) for the case  $m = n$ , we conclude that  $C$  is closed and therefore:

$$C = C(p) = (-1)^n c_p^{\mathfrak{g}}(x_1, \dots, x_{n+1}) - \varsigma(n+1) \iota(v_1 \wedge \dots \wedge v_{n+1}) \omega|_p = 0.$$

Hence, the collection  $f_1, \dots, f_{n-1}, \tilde{f}_n$  gives the desired morphism, and it follows from the definition of  $f_1$  that the diagram (34) commutes.  $\square$

A more homotopy-theoretic conceptual interpretation of the above proposition appears in [12]. Below we give some examples of what kinds of Lie  $n$ -algebras can be constructed in this way.

**Example 9.12** (Heisenberg  $n$ -algebra). Let  $V$  be a finite-dimensional real vector space. A linear non-zero skew-symmetric form  $\omega \in \Lambda^{n+1} V^*$  of degree  $n+1$  induces a translation-invariant closed differential form on  $V$ . Therefore,  $(V, \omega)$  is a pre- $n$ -plectic manifold and  $V$  (seen as an abelian Lie algebra) acts on itself via translations. This gives a Lie algebra morphism  $v_-: V \rightarrow \mathfrak{X}_{\text{Ham}}(V)$ . Since  $\omega$  is non-zero, the degree  $(n+1)$  class  $[c_p^V]$  is non-trivial. Hence, there is no homotopy moment map lifting the action of  $V$ . Let  $\widehat{V}$  be the associated central  $n$ -extension. Prop. 9.11 implies that  $\widehat{V}$  sits in a commuting diagram of  $L_\infty$ -algebras of the form (34).

Compare with Ex. 8.3, for which any linear action on  $(V, \omega)$  admits a moment map.

**Example 9.13** (String Lie 2-algebra). Let  $G$  be a compact connected simple Lie group, and let  $\omega = \frac{1}{12} \langle \theta_L, [\theta_L, \theta_L] \rangle$  be the Cartan 3-form. As previously mentioned in Sec. 8.3,  $(G, \omega)$  is a 2-plectic manifold, and the action of  $G$  on itself via conjugation lifts to a homotopy moment map. Clearly,  $\omega$  is also preserved by the action of  $G$  on itself via left-translation, but the corresponding degree 3 class  $[c_p^{\mathfrak{gl}}]$  is not trivial. (Indeed,  $\langle \cdot, [\cdot, \cdot] \rangle$  is a generator of  $H_{\text{CE}}^3(\mathfrak{g})$ ). The corresponding central 2-extension is the **string Lie 2-algebra**  $\text{string}(\mathfrak{g})$ . When  $G = \text{Spin}(n)$ , this Lie 2-algebra (or rather its integration) plays a very interesting role in a certain elliptic cohomology theory and in the theory of “spin structures” on loop spaces. (See, for example, Sec. 1 of [29] for a review.)

Since  $G$  is compact and simple we have  $H_{\text{dR}}^1(G) \cong H_{\text{CE}}^1(\mathfrak{g}) = 0$ . Hence, Prop. 9.11 implies that there is a commuting diagram of  $L_\infty$ -algebras:

$$\begin{array}{ccc} \text{string}(\mathfrak{g}) & \longrightarrow & \text{PoissonLie}(M, \omega) \\ \pi_{\mathfrak{g}} \downarrow & & \downarrow \pi \\ \mathfrak{g} & \xrightarrow{v_-^{\text{left}}} & \mathfrak{X}_{\text{Ham}}(M) \end{array}$$

This result gives a nice conceptual interpretation to the relationship previously established in [4] between  $\text{string}(\mathfrak{g})$  and  $L_\infty(G, \omega)$ .

## 10. MODULI SPACES OF FLAT CONNECTIONS

Here we consider homotopy moment maps on spaces of connections over higher-dimensional manifolds (see Thm. 10.7). Currently, our motivation for this example is simply to generalize the famous Atiyah-Bott construction [2] in symplectic geometry. Since our construction begins by considering an invariant polynomial in  $S(\mathfrak{g}^*)^G$  of higher degree  $\geq 2$ , it is possible that some of these ideas could find application in certain topological field theories which generalize Chern-Simons theory.

**10.1. Invariant polynomials.** Given an integer  $n \geq 1$ , we consider the following data:

- a real, finite dimensional Lie algebra  $\mathfrak{g}$  equipped with a invariant polynomial  $q \in S^{n+1}(\mathfrak{g}^*)^G$ ,
- a  $(n+1)$ -dimensional compact, oriented manifold  $M$ , and
- a principal  $G$ -bundle  $\pi: P \rightarrow M$ , where  $G$  is any Lie group integrating  $\mathfrak{g}$ . The group  $G$  acts on the right of  $P$  via diffeomorphisms  $R_g$ .

We denote by

$$\hat{\xi}(p) = \frac{d}{dt} R_{\exp(t\xi)}(p)|_{t=0}$$

the infinitesimal generators of the action of  $G$  on  $P$ , for all  $\xi \in \mathfrak{g}$ . We say an invariant polynomial  $q$  is **non-degenerate** iff the map

$$\begin{aligned} \mathfrak{g} &\rightarrow S^n(\mathfrak{g}^*) \\ x &\mapsto \iota_x q \end{aligned}$$

is injective.

**Example 10.1.** If  $G$  is a matrix group, then the symmetrized (real) trace gives obvious examples of invariant polynomials. In particular, for  $G = \mathrm{SU}(N)$ , we define:

$$q_k(x_1, \dots, x_k) = -\frac{1}{k!} \sum_{\sigma \in \mathcal{S}_k} \mathrm{Re} \, \mathrm{Tr}(x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(k)}) \quad \forall x_i \in \mathfrak{su}(N).$$

It is well known that the polynomial  $q_2$  gives a real inner product on  $\mathfrak{su}(N)$ , but more generally, one can show for  $G = \mathrm{SU}(2)$  that every  $q_{2n}$  is non degenerate, for  $n > 0$ . Consider  $\{e_i\}$  the basis of  $G = \mathrm{SU}(2)$  given by

$$e_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad e_2 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad e_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The identity

$$(35) \quad e_i e_j = -\frac{1}{4} \delta_{ij} I + \frac{1}{2} \sum_k \varepsilon_{ijk} e_k,$$

where  $\varepsilon_{ijk}$  is totally skew and  $\varepsilon_{123} = 1$ , implies that  $e_i e_j^{2n} = (-\frac{1}{4})^n e_i$ . Therefore

$$e_i e_j^{2n+1} = (-\frac{1}{4})^{n+1} \delta_{ij} I + \frac{1}{2} (-\frac{1}{4})^n \sum_k \varepsilon_{ijk} e_k.$$

In particular

$$(36) \quad q_{2(n+1)}(e_i, e_j, \dots, e_j) = -\mathrm{Re} \, \mathrm{Tr}(e_i e_j^{2n+1}) = -2(-\frac{1}{4})^{n+1} \delta_{ij}.$$

This enables to show that  $q_{2(n+1)}$  is non-degenerate. Indeed, suppose there exists  $x \in \mathfrak{su}(2)$  such that

$$q_{2(n+1)}(x, y_1, \dots, y_{2n+1}) = 0 \quad \forall y_i \in \mathfrak{su}(2).$$

Write  $x = \sum_i x^i e_i$ , then (36) means that  $x_i = q_{2(n+1)}(x, e_i, \dots, e_i) = 0$  for all  $i$  implies that  $x = 0$ . So  $q_{2(n+1)}$  is non-degenerate. Note that this is not true for all  $q_k$ . In fact, Eq. (35) shows that  $q_3 = 0$ .



**10.2. The gauge group action.** A **connection** on  $P$  is a  $\mathfrak{g}$ -valued 1-form  $A \in \Omega^1(P, \mathfrak{g})$  satisfying

$$A(\hat{\xi}) = \xi, \quad R_g^* A = \text{Ad}_{g^{-1}} A$$

for all  $\xi \in \mathfrak{g}$  and  $g \in G$ . The set  $\mathcal{A}$  of all connections on  $P$  is an affine space modeled on the vector space  $(\Omega_{\text{hor}}^1(P) \otimes \mathfrak{g})^G$ , where the first factor denotes the 1-forms on  $P$  annihilated by vectors tangent to the fibers. The left action of  $G$  on  $\Omega_{\text{hor}}^1(P) \otimes \mathfrak{g}$  is

$$g \cdot (\alpha \otimes \xi) = R_g^* \alpha \otimes \text{Ad}_g \xi.$$

The **gauge group**  $\mathcal{G}$  of  $P$  is the group of smooth maps  $f: P \rightarrow G$  satisfying

$$R_g^* f(p) = g^{-1} f(p) g \quad \forall g \in G.$$

Such a map  $f$  can be identified with a  $G$ -equivariant map  $\phi: P \rightarrow P$  covering  $\text{id}_M$  where

$$\phi(p) = R_{f(p)}(p).$$

The gauge group acts on the space of connections  $\mathcal{A}$  from the left:

$$(37) \quad f \cdot A = \text{Ad}_f A + (f^{-1})^* \theta_L,$$

where  $\theta_L \in \Omega^1(G, \mathfrak{g})$  is the left invariant Maurer-Cartan form on  $G$  and  $f^{-1}$  is the composition of  $f$  with the inversion on  $G$ . If  $\phi$  is the bundle automorphism associated to  $f$ , then the action is simply  $\phi \cdot A = (\phi^{-1})^* A$ .

To obtain the infinitesimal analog of the above, we consider maps  $X: P \rightarrow \mathfrak{g}$  satisfying

$$R_g^* X(p) = \text{Ad}_{g^{-1}} X(p) \quad \forall g \in G.$$

The space of all such maps forms the **Lie algebra of infinitesimal gauge transformations**  $\text{Lie}(\mathcal{G})$ . This plays the role of the Lie algebra associated to  $\mathcal{G}$ .

**Remark 10.2.** Indeed,  $\mathcal{G}$  is a locally exponential Lie group modeled on the Lie algebra  $\text{Lie}(\mathcal{G})$  [33, Thm. 1.11]. This means that for each  $X \in \text{Lie}(\mathcal{G})$  the initial value problem

$$\gamma(0) = e_G, \quad \gamma(t)^{-1} \cdot \gamma'(t) = X$$

has a solution  $\gamma_X \in C^\infty(\mathbb{R}, \mathcal{G})$ , and there exists a unique smooth exponential map

$$\exp: \text{Lie}(\mathcal{G}) \rightarrow \mathcal{G}, \quad X \mapsto \gamma_X(1)$$

and an open neighborhood  $0 \in W \subset \text{Lie}(\mathcal{G})$  such that  $\exp|_W$  is a diffeomorphism onto some open neighborhood of the identity  $e_G$ .

Differentiating the action (37) gives an action of  $\text{Lie}(\mathcal{G})$  on  $\mathcal{A}$ . Specifically, given  $X \in \text{Lie}(\mathcal{G})$  we define the fundamental vector field  $V_X: \mathcal{A} \rightarrow (\Omega_{\text{hor}}^1(P) \otimes \mathfrak{g})^G$  by

$$(38) \quad V_X(A) = \frac{d}{dt} (\exp(-tX) \cdot A) |_{t=0}.$$

Note that the assignment  $X \mapsto V_X$  is a Lie algebra morphism from  $\text{Lie}(\mathcal{G})$  to  $\mathfrak{X}(\mathcal{A})$ . Also, a simple calculation shows

$$V_X(A) = d_A X.$$

where  $d_A: (\Omega_{\text{hor}}^\bullet(P) \otimes \mathfrak{g})^G \rightarrow (\Omega_{\text{hor}}^{\bullet+1}(P) \otimes \mathfrak{g})^G$  is the **covariant derivative**

$$d_A \alpha = d\alpha + [A, \alpha].$$

**10.3. Closed forms from invariant polynomials.** An invariant polynomial  $q \in S^{n+1}(\mathfrak{g}^*)^G$  gives a constant, hence closed,  $(n+1)$ -form on the space of connections  $\mathcal{A}$ . To see this, first consider the following  $(n+1)$ -form on  $P$ :

$$q(\alpha_1, \dots, \alpha_{n+1}) \in \Omega^{n+1}(P),$$

where each  $\alpha_i$  is in  $(\Omega_{\text{hor}}^1(P) \otimes \mathfrak{g})^G$ . This form clearly vanishes when contracted with any vertical vector on  $P$ . Moreover, the Ad invariance of  $q$  combined with the  $G$  invariance of  $\alpha_i$  implies that  $q(\alpha_1, \dots, \alpha_{n+1})$  is invariant under the action of  $G$  on  $P$ . Therefore,  $q(\alpha_1, \dots, \alpha_{n+1})$  is **basic** i.e. it corresponds to the pullback of a unique  $(n+1)$ -form on  $M$  along  $\pi: P \rightarrow M$ . We “abuse notation” by also denoting this  $(n+1)$ -form on  $M$  as  $q(\alpha_1, \dots, \alpha_{n+1})$ . By integration, we then obtain a closed  $(n+1)$ -form on  $\mathcal{A}$ :

$$(39) \quad \omega(\alpha_1, \dots, \alpha_{n+1})|_A = \int_M q(\alpha_1, \dots, \alpha_{n+1}) \quad \forall \alpha_i \in T_A \mathcal{A} = (\Omega_{\text{hor}}^1(P) \otimes \mathfrak{g})^G.$$

The following proposition shows that, for some cases,  $\omega$  is in fact  $n$ -plectic.

**Proposition 10.3.** *If the invariant polynomial  $q \in S^{n+1}(\mathfrak{g}^*)^G$  is non-degenerate, then  $\omega$  (39) is an  $n$ -plectic structure on  $\mathcal{A}$ .*

*Proof.* Given  $\beta \in (\Omega_{\text{hor}}^1(P) \otimes \mathfrak{g})^G$  such that

$$\int_M q(\beta, \alpha_2, \dots, \alpha_{n+1}) = 0 \quad \forall \alpha_i \in (\Omega_{\text{hor}}^1(P) \otimes \mathfrak{g})^G,$$

we assume, in order to lead to a contradiction, that there exists  $p \in P$  such that  $\beta|_p \neq 0$ . Let  $U \subseteq M$  be a chart containing  $y = \pi(p)$  admitting a trivialization  $\tau: \pi^{-1}(U) \xrightarrow{\sim} U \times G$  such that  $\tau(p) = (y, e)$ . Let  $x^1, \dots, x^{n+1}$  be coordinates on  $U$ . Working locally over  $\pi^{-1}(U)$ , and implicitly using the trivialization, we write  $\beta = \sum_{i=1}^{n+1} \beta_i d\pi^* x^i$  where  $\beta_i: U \times G \rightarrow \mathfrak{g}$  satisfies  $\beta_i(x, g) = \text{Ad}_{g^{-1}} \beta_i(x, e)$ . By our assumption, there exists an  $i$  such that  $\beta_i(y, e) \neq 0$ . Without loss of generality, we may further assume  $i = 1$ .

Since  $q$  is non-degenerate, there exists  $\xi_2, \dots, \xi_{n+1} \in \mathfrak{g}$  such that  $q(\beta_1(y, e), \xi_2, \dots, \xi_{n+1}) > 0$ . Hence, there exists a smaller neighborhood  $V \subseteq U$  containing  $y$  such that

$$q(\beta_1(x, e), \xi_2, \dots, \xi_{n+1}) > 0 \quad \forall x \in V.$$

Define  $\mathfrak{g}$ -valued maps  $f_2, \dots, f_{n+1}$  on  $\pi^{-1}(V)$  by

$$f_i(x, g) = \text{Ad}_{g^{-1}} \xi_i.$$

Finally, let  $\varphi: M \rightarrow [0, 1]$  be a “bump function” whose support is contained in  $V$ .

Using all of this, we can define global  $\mathfrak{g}$ -valued 1-forms  $\alpha_2, \dots, \alpha_{n+1}$  on  $P$  by

$$\alpha_i = \pi^*(\varphi) d\pi^*(x^i) \otimes f_i.$$

By construction, each  $\alpha_i$  is in  $(\Omega_{\text{hor}}^1(P) \otimes \mathfrak{g})^G$ , and therefore we have a contradiction:

$$0 = \int_M q(\beta, \alpha_2, \dots, \alpha_{n+1}) = \int_{\text{supp } \varphi} q(\beta_1(x, e), \xi_2, \dots, \xi_{n+1}) dx^1 dx^2 \dots dx^{n+1} > 0$$

This implies that  $\beta$  is zero. Hence,  $\omega$  is non-degenerate. □

**10.4. The moment map.** From here on, we assume the following:

- The principal  $G$ -bundle  $P \rightarrow M$  admits a flat connection.

We equip the space of connections  $\mathcal{A}$  with the closed  $(n+1)$ -form  $\omega$  given in Eq. (39).

We begin by considering the linear map  $\mu: \text{Lie}(\mathcal{G}) \rightarrow \Omega^{n-1}(\mathcal{A})$  defined as

$$(40) \quad \mu(X)(\alpha_1, \dots, \alpha_{n-1})|_A = \int_M q(F_A, \alpha_1, \dots, \alpha_{n-1}, X),$$

for all  $X \in \text{Lie}(\mathcal{G})$ ,  $A \in \mathcal{A}$  and  $\alpha_i \in T_A\mathcal{A}$ . Here  $F_A$  is the curvature of the connection  $A$ :

$$F_A = dA + \frac{1}{2}[A, A] \in (\Omega_{\text{hor}}^2(P) \otimes \mathfrak{g})^G.$$

The curvature is horizontal and it satisfies  $R_g^* F_A = \text{Ad}_{g^{-1}} F_A$ . So, the  $(n+1)$ -form  $q(F_A, \alpha_1, \dots, \alpha_{n-1}, X)$  on  $P$  descends to a unique  $(n+1)$ -form on  $M$ .

**Proposition 10.4.** *The map  $\mu: \text{Lie}(\mathcal{G}) \rightarrow \Omega^{n-1}(\mathcal{A})$  defined by Eq. (40) is  $\mathcal{G}$ -equivariant, i.e.*

$$(f^{-1})^* \mu(X) = \mu(\text{Ad}_f X) \quad \forall f \in \mathcal{G}, X \in \text{Lie}(\mathcal{G}).$$

*Proof.* Let  $A \in \mathcal{A}$  and  $\alpha_1, \dots, \alpha_{n-1} \in T_A\mathcal{A}$ . We have

$$(41) \quad (f^* \mu(X))(\alpha_1, \dots, \alpha_{n-1})|_A = \mu(X)(f_* \alpha_1, \dots, f_* \alpha_{n-1})|_{f \cdot A} = \int_M q(F_{f \cdot A}, f_* \alpha_1, \dots, f_* \alpha_{n-1}, X).$$

A straightforward computation shows that  $F_{f \cdot A} = \text{Ad}_f F_A$ . It is also not difficult to show that the differential  $f_*$  of the map  $A \mapsto f \cdot A$  is  $f_* \alpha_i = \text{Ad}_f \alpha_i$ . Hence, from the  $\text{Ad}_f$ -invariance of  $q$ , we see that the right-hand side of Eq. (41) is

$$\mu(\text{Ad}_{f^{-1}} X)(\alpha_1, \dots, \alpha_{n-1})|_A$$

□

Next, we show the image of  $\mu$  lies in Hamiltonian forms. The associated Hamiltonian vector fields are those induced by the infinitesimal gauge transformations (38).

**Proposition 10.5.** *If  $X \in \text{Lie}(\mathcal{G})$ , then  $\mu(X)$  is a Hamiltonian  $(n-1)$ -form with Hamiltonian vector field  $V_X$ , where*

$$V_X|_A = d_A X = dX + [A, X] \quad \forall A \in \mathcal{A}.$$

To prove this, we will use the following lemma.

**Lemma 10.6.** *If  $q \in S^k(\mathfrak{g}^*)^G$  is a degree  $k$  invariant polynomial,  $A \in \mathcal{A}$  is a connection, and  $\beta_1, \dots, \beta_k \in (\Omega_{\text{hor}}^\bullet(P) \otimes \mathfrak{g})^G$  are forms with  $|\beta_1| + |\beta_2| + \dots + |\beta_k| = n$ , then*

$$\sum_{i=1}^k (-1)^{|\beta_1| + \dots + |\beta_{i-1}|} \int_M q(\beta_1, \dots, d_A \beta_i, \dots, \beta_k) = 0,$$

where the above sign for  $i=1$  is defined to be  $+1$ .

*Proof.* By replacing  $d_A \beta_i$  by  $d\beta_i + [A, \beta_i]$  for all  $i$  in

$$(42) \quad \sum_{i=1}^k (-1)^{|\beta_1| + \dots + |\beta_{i-1}|} q(\beta_1, \dots, d_A \beta_i, \dots, \beta_k)$$

we rewrite the above as the sum of two basic  $(n+1)$ -forms on  $P$ :

$$d(q(\beta_1, \beta_2, \dots, \beta_k)) + \left( \sum_{i=1}^k (-1)^{|\beta_1| + \dots + |\beta_{i-1}|} q(\beta_1, \dots, [A, \beta_i], \dots, \beta_k) \right),$$

where the summation over  $i$  is, in fact, zero by the infinitesimal  $G$ -invariance of  $q$ . Hence, (42) descends to an exact  $(n+1)$ -form on  $M$ , and so its integral vanishes by Stokes' theorem. □

*Proof of Prop. 10.5.* We need to show  $d\mu(X) = -\iota(V_X)\omega$  for all  $X \in \text{Lie}(\mathcal{G})$ . Let  $A \in \mathcal{A}$ . Given tangent vectors  $\alpha_1, \dots, \alpha_n \in T_A\mathcal{A}$ , we denote by the same symbols their extension to constant vector fields on  $\mathcal{A}$ . Hence,  $[\alpha_i, \alpha_j] = 0$  for all  $i$  and  $j$ , and so the de Rham differential becomes

$$d\mu(X)(\alpha_1, \dots, \alpha_n) = \sum_i (-1)^{i+1} \mathcal{L}_{\alpha_i}(\mu(X)(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_n)).$$

The identity

$$\left. \frac{d}{dt} F_{A+t\alpha_i} \right|_{t=0} = d_A \alpha_i,$$

combined with Lemma 10.6 imply that

$$\begin{aligned} d\mu(X)(\alpha_1, \dots, \alpha_n) &= \sum_i (-1)^{i+1} \int_M q(d_A \alpha_i, \alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_n, X) \\ &= -(-1)^n \int_M q(\alpha_1, \dots, \alpha_n, d_A X) \\ &= -(\iota(V_X)\omega)(\alpha_1, \dots, \alpha_n). \end{aligned}$$

□

The main result of this section is the following theorem. Its proof features the obstruction theory developed in Sec. 9.

**Theorem 10.7.** *There exists a homotopy moment map*

$$(f_i): \text{Lie}(\mathcal{G}) \rightarrow L_\infty(\mathcal{A}, \omega)$$

*lifting the action of  $\text{Lie}(\mathcal{G})$  on  $(\mathcal{A}, \omega)$  such that*

$$f_1(X)(\alpha_1, \dots, \alpha_{n-1})|_A = \mu(X)(\alpha_1, \dots, \alpha_{n-1})|_A = \int_M q(F_A, \alpha_1, \dots, \alpha_{n-1}, X)$$

*for all  $A \in \mathcal{A}$  and  $\alpha_i \in T_A\mathcal{A}$ .*

*Proof.* By Remark 9.8, the theorem is proved if we can show that the assumptions listed in Thm. 9.7 are satisfied by taking the linear map  $\phi$  (defined there) to be  $\mu$ . Let  $A_0 \in \mathcal{A}$  be a flat connection. We identify  $\mathcal{A}$  with the locally convex topological vector space  $(\Omega_{\text{hor}}^1(P) \otimes \mathfrak{g})^G$  so that  $A_0$  corresponds to the origin. The Poincaré Lemma holds [5, Lem. 1.4.1], and hence the de Rham cohomology of  $\mathcal{A}$  is  $\mathbb{R}$  in degree 0, and trivial in all higher degrees. Next, observe that the Lie algebra cocycle

$$c_{A_0}^{\text{Lie}(\mathcal{G})}(X_1, \dots, X_{n+1}) = \pm \omega(V_{X_1}, \dots, V_{X_{n+1}})|_{A_0} = \int_M q(d_{A_0} X_1, \dots, d_{A_0} X_{n+1}).$$

introduced in Cor. 9.4 is trivial. Indeed, Lemma 10.6 implies that

$$\int_M q(d_{A_0} X_1, \dots, d_{A_0} X_{n+1}) = \sum_{i=2}^{n+1} \pm \int_M q(X_1, \dots, d_{A_0}^2 X_i, \dots, d_{A_0} X_{n+1}),$$

and, since  $A_0$  is flat, we have  $d_{A_0}^2 X_i = [F_{A_0}, X_i] = 0$ . □

**10.5. Reduction.** We equip  $(\mathcal{A}, \omega)$  with a moment map given by Thm. 10.7, and now describe a type of Marsden-Weinstein reduction. As we shall see, when  $\omega$  is non-degenerate, the quotient of a zero-level set gives a pre- $n$ -plectic structure on the moduli space of flat connections.

We denote by  $\mathcal{A}_{\text{flat}} \subset \mathcal{A}$  the set of flat connections. If  $A$  is a flat connection, then the tangent space  $T_A\mathcal{A}_{\text{flat}}$  consists of all vectors  $\alpha \in T_A\mathcal{A}$  that satisfy  $d_A \alpha = d\alpha + [A, \alpha] = 0$ . We also consider the following “zero level set” of  $\mu$ :

$$\mathcal{C} = \{A \in \mathcal{A} \mid \mu(X)|_A = 0 \text{ for all } X \in \text{Lie}(\mathcal{G})\}.$$

The  $\mathcal{G}$ -action restricts to  $\mathcal{A}_{\text{flat}}$ , and by Prop. 10.4, it also restricts to  $\mathcal{C}$ . It follows from the definition of  $\mu$  that

$$\mathcal{A}_{\text{flat}} \subseteq \mathcal{C}.$$

At least in certain cases, the two subspaces are equal, as the following proposition demonstrates.

**Proposition 10.8.** *Let  $q \in S^{n+1}(\mathfrak{g}^*)^G$  be the invariant polynomial used in constructing  $\omega \in \Omega^{n+1}(\mathcal{A})$ . If  $q$  is non-degenerate, then  $\mathcal{A}_{\text{flat}} = \mathcal{C}$ .*

*Proof.* Let  $A \in \mathcal{C}$  and assume, in order to lead to a contradiction, that there exists  $p \in P$  such that  $F_A|_p \neq 0$ . We proceed as we did in the proof of Prop. 10.3, replacing there the 1-form  $\beta$  with the 2-form  $F_A$ . The non-degeneracy allows us to construct 1-forms  $\alpha_1, \dots, \alpha_{n-1} \in (\Omega_{\text{hor}}^1(P) \otimes \mathfrak{g})^G$  and element  $X \in \text{Lie}(\mathcal{G})$  such that

$$\int_M q(F_A, \alpha_1, \dots, \alpha_{n-1}, X) > 0.$$

Hence, we have  $\mu(X)(\alpha_1, \dots, \alpha_{n-1})|_A > 0$ , which contradicts  $A \in \mathcal{C}$ .  $\square$

Next, we consider pre- $n$ -plectic structures induced on  $\mathcal{A}_{\text{flat}}$  and  $\mathcal{C}$ .

**Proposition 10.9.** *If  $\iota: \mathcal{A}_{\text{flat}} \hookrightarrow \mathcal{A}$  is the inclusion, then*

$$V_X \in \ker \iota^* \omega$$

*for all  $X \in \text{Lie}(\mathcal{G})$ .*

*Proof.* Let  $X \in \text{Lie}(\mathcal{G})$  and  $A \in \mathcal{A}_{\text{flat}}$ . Fix  $\alpha_2, \dots, \alpha_{n+1} \in T_A \mathcal{A}_{\text{flat}}$ . We have the equalities

$$\omega(V_X, \alpha_2, \dots, \alpha_{n+1})|_A = \omega(d_A X, \alpha_2, \dots, \alpha_{n+1}) = \int_M q(d_A X, \alpha_2, \dots, \alpha_{n+1}).$$

Hence, Lemma 10.6 implies that

$$\omega(V_X, \alpha_2, \dots, \alpha_{n+1})|_A = \sum_{i=2}^{n+1} \pm \int_M q(X, \dots, d_A \alpha_i, \dots, \alpha_{n+1}).$$

The right-hand side of the above is zero, since  $d_A \alpha_i = 0$  for all  $\alpha_i$ .  $\square$

Recall that Prop. 10.5 implies that  $\iota^* \omega$  is  $\mathcal{G}$ -invariant. This fact combined with the above two propositions gives the following:

**Corollary 10.10.** *If the quotient space  $\mathcal{A}_{\text{flat}}/\mathcal{G}$  is a smooth manifold, then it carries a closed  $(n+1)$ -form induced by  $\omega$ . In addition, if  $\omega$  is non-degenerate, then  $\mathcal{A}_{\text{flat}}/\mathcal{G} = \mathcal{C}/\mathcal{G}$ .*

Unfortunately, if  $\omega$  is non-degenerate, then it does *not* follow that the  $(n+1)$ -form on  $\mathcal{C}/\mathcal{G}$  is non-degenerate, as this example shows.

**Example 10.11.** We consider the simplest possible “higher” case. Let  $M$  be of dimension 3,  $G = \mathbb{R}$ , and  $P$  the trivial bundle  $\mathbb{R} \times M \rightarrow M$ . Let  $p: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be just the multiplication of three numbers; it is clearly a non-degenerate invariant polynomial on the Lie algebra.

Since the bundle is trivial, we can write our data as

- $\mathcal{A} = \Omega^1(M)$ , a vector space
- $\mathcal{G} = C^\infty(M)$
- $\text{Lie}(\mathcal{G}) = C^\infty(M)$
- $V_X = dX$  for all  $X \in \text{Lie}(\mathcal{G})$
- $\omega(\alpha_1, \alpha_2, \alpha_3)|_A = \int_M \alpha_1 \wedge \alpha_2 \wedge \alpha_3$

Clearly  $\mathcal{A}_{flat} = \Omega_{closed}^1(M)$ , again a vector space, and  $\mathcal{A}_{flat}/\mathcal{G} = H^1(M)$ .

The induced 3-form on  $H^1(M)$  is the just the evaluation on the fundamental class of  $M$ :

$$H^1(M) \otimes H^1(M) \otimes H^1(M) \mapsto \mathbb{R}, \quad a_1 \otimes a_2 \otimes a_3 \mapsto \langle a_1 \wedge a_2 \wedge a_3, [M] \rangle.$$

Therefore it is non-degenerate iff

for all  $a_1 \in H^1(M)$ ,

$$(43) \quad a_1 a_2 a_3 = 0 \text{ for all } a_2, a_3 \in H^1(M) \Rightarrow a_1 = 0.$$

In general, condition (43) is not satisfied. Recall that the pairing  $H^1(M) \otimes H^2(M) \rightarrow H^3(M) \cong \mathbb{R}$  is non-singular, by Poincaré duality. Hence, the non-degeneracy condition (43) is satisfied iff the product map

$$H^1(M) \otimes H^1(M) \rightarrow H^2(M)$$

is surjective.

So, for example, if  $M = S^2 \times S^1$ , then condition (43) is not satisfied, and indeed  $H^1(S^2 \times S^1) \cong \mathbb{R}$  must carry the zero 3-form. When  $M = S^1 \times S^1 \times S^1$  is the torus, condition (43) is satisfied, and  $H^1(M) \cong \mathbb{R}^3$  carries a constant volume form.

## 11. LOOP SPACES

In this section, we show how homotopy moment maps for  $G$  actions on pre-2-plectic manifolds  $(M, \omega)$  can be transgressed to ordinary moment maps on the associated pre-symplectic loop space  $LM$  (see Thm. 11.2). Motivation for studying such actions arises, for example, in topological field theory. There one can consider a group of symmetries  $G$  acting on a “target space”  $M$  equipped with a closed form  $\omega$ . The form can be transgressed to a mapping space  $\text{Map}(X, M)$  i.e. the “space of fields”. The induced  $G$  action is then defined “point-wise”. The results of this section could be interpreted as an elementary example of this process for the case  $X = S^1$ . Roughly speaking, this demonstrates how the higher symplectic geometry on  $M$  can interact with the ordinary geometry on  $\text{Map}(X, M)$ .

**11.1. Actions on loop spaces.** For any manifold  $M$ , the free loop space  $LM$ , i.e. the space of smooth loops

$$LM = C^\infty(S^1, M),$$

is an infinite-dimensional Fréchet manifold ([5], [31] and [22]). The tangent space at  $\gamma \in LM$  can be identified with global sections of the pullback of  $TM$  along  $\gamma: S^1 \rightarrow M$ , i.e.

$$T_\gamma LM = C^\infty(S^1, \gamma^* TM)$$

There is a degree  $-1$  chain map

$$\ell: \Omega^\bullet(M) \rightarrow \Omega^{\bullet-1}(LM)$$

called **transgression**. Explicitly, it sends a  $k$ -form  $\alpha$  on  $M$  to the  $(k-1)$ -form  $\alpha^\ell$  on  $LM$  given by the formula

$$\alpha^\ell|_\gamma(v_1, \dots, v_{k-1}) = \int_0^{2\pi} \alpha(v_1, \dots, v_{k-1}, \dot{\gamma})|_{\gamma(s)} ds \quad \forall \gamma \in LM, \quad \forall v_1, \dots, v_{k-1} \in T_\gamma LM.$$

Since transgression commutes with the de Rham differential, any pre- $n$ -plectic structure  $\omega$  on  $M$  gives a pre- $(n-1)$ -plectic structure  $\omega^\ell$  on  $LM$ .

Suppose  $M$  is a manifold equipped with a  $G$ -action  $G \times M \rightarrow M$ . This induces a “point-wise” action  $G \times LM \rightarrow LM$  given by:

$$(g \cdot \gamma)(s) = g \cdot \gamma(s) \quad \forall g \in G, \quad \forall \gamma \in LM.$$

Given an element  $x \in \mathfrak{g}$  and a loop  $\gamma \in LM$ , we obtain a smooth path in  $LM$

$$\mathbb{R} \ni t \mapsto \exp(-xt) \cdot \gamma,$$

and an action of  $\mathfrak{g}$  on  $LM$  via the fundamental vector field

$$v_x^\ell|_\gamma = \left. \frac{d}{dt} \exp(-xt) \cdot \gamma \right|_{t=0} \quad \forall \gamma \in LM.$$

We then observe, by differentiation, that the fundamental vector field  $v_x^\ell$  on  $LM$  evaluated at a point  $\gamma \in LM$  is just  $\gamma^* v_x$  (a section of  $\gamma^* TM \rightarrow S^1$ ), where  $v_x$  is the fundamental vector field on  $M$  associated to  $x$ .

More generally, restricting vector fields on  $M$  to loops in  $M$  gives us a map

$$\begin{aligned} \Gamma(TM) &\rightarrow \Gamma(TLM) \\ v &\mapsto v^\ell, \end{aligned}$$

defined by  $v^\ell|_\gamma = \gamma^* v$ . That  $v^\ell$  is a smooth vector field with respect to the induced smooth structure on  $TLM$  follows from the fact that it is the composition of two smooth maps. The first of these is a map  $LM \rightarrow LTM$  given by  $\gamma \mapsto v \circ \gamma$  [31, Thm. 3.27]. The second map is the natural diffeomorphism of vector bundles  $LTM \cong TLM$  covering the identity on  $LM$  [31, Thm. 4.2].

**11.2. Actions on pre-symplectic loop spaces.** Suppose  $(M, \omega)$  is a pre-2-plectic manifold. Then  $(LM, \omega^\ell)$  is a pre-symplectic manifold. We have:

**Proposition 11.1.** *If  $\alpha$  is a Hamiltonian 1-form with Hamiltonian vector field  $v$ , then the vector field  $v^\ell$  is Hamiltonian for the function  $\alpha^\ell: LM \rightarrow \mathbb{R}$ .*

*Proof.* By assumption we have  $d\alpha = -\iota(v)\omega$ , and we wish to show

$$d\alpha^\ell = -\iota(v^\ell)\omega^\ell.$$

Let  $\gamma \in LM$ . For all  $u \in T_\gamma LM$ , we have

$$\begin{aligned} (\iota_{v^\ell} \omega^\ell)(u) &= \int_0^{2\pi} \omega(v^\ell, u, \dot{\gamma})|_{\gamma(s)} ds \\ &= - \int_0^{2\pi} d\alpha(u, \dot{\gamma})|_{\gamma(s)} ds \\ &= -(d\alpha)^\ell(u) \end{aligned}$$

□

Now suppose that  $(M, \omega)$  is equipped with a  $G$  action and with a homotopy moment map  $\mathfrak{g} \rightarrow L_\infty(M, \omega)$ . Let  $f_1: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^1(M)$ , and  $f_2: \mathfrak{g} \otimes \mathfrak{g} \rightarrow C^\infty(M)$  be the corresponding structure maps for this moment map. For  $x \in \mathfrak{g}$ , the vector field  $v_x$  is Hamiltonian for the 1-form  $\alpha_x = f_1(x)$ . The 2-form  $\omega^\ell$  on  $LM$  is  $G$ -invariant, and the  $v_x^\ell$  are Hamiltonian. The next theorem says that the homotopy moment map on  $(M, \omega)$  transgresses to an ordinary moment map for the action of  $G$  on  $(LM, \omega^\ell)$

**Theorem 11.2.** *If  $(M, \omega)$  is a pre-2-plectic manifold equipped with a  $G$  action and a homotopy moment map  $f: \mathfrak{g} \rightarrow L_\infty(M, \omega)$ , then*

$$\begin{aligned} \psi: \mathfrak{g} &\rightarrow C^\infty(LM) \\ x &\mapsto (f_1(x))^\ell \end{aligned}$$

*is a moment map for the induced action of  $G$  on the pre-symplectic loop space  $(LM, \omega^\ell)$*

*Proof.* To prove the theorem, it is sufficient to show that  $\psi$  is a Lie algebra morphism.



The bracket of the functions  $\psi(x)$  and  $\psi(y)$  evaluated at  $\gamma \in LM$  is:

$$\begin{aligned} \{\psi(x), \psi(y)\} |_\gamma &= \int_0^{2\pi} \omega(v_x, v_y, \dot{\gamma})|_{\gamma(s)} ds \\ &= \int_0^{2\pi} \iota_{\dot{\gamma}} l_2(f_1(x), f_1(y))|_{\gamma(s)} ds \\ &= (l_2(f_1(x), f_1(y)))^\ell |_\gamma, \end{aligned}$$

where  $l_2$  is the bi-linear bracket for the Lie 2-algebra  $L_\infty(M, \omega)$ . On the other hand,  $\psi([x, y])$  is, by definition, equal to the transgression of the 1-form  $f_1([x, y])$ . The definition of an  $L_\infty$ -morphism implies that

$$(44) \quad l_2(f_1(x), f_1(y)) - f_1([x, y]) = df_2(x, y).$$

Since  $df_2(x, y)$  is an exact 1-form, Stokes theorem implies for all  $\gamma \in LM$ :

$$(d(f_2(x, y)))^\ell = \int_0^{2\pi} \iota_{\dot{\gamma}} d(f_2(x, y))|_{\gamma(s)} ds = 0$$

Hence applying the transgression operator  $\ell$  to Eq. (44) we obtain

$$\{\psi(x), \psi(y)\} - \psi([x, y]) = 0.$$

□

**Remark 11.3.** The map

$$\Omega_{\text{Ham}}^1(M) \rightarrow C_{\text{Ham}}^\infty(LM), \quad \alpha \mapsto \alpha^\ell$$

is well-defined by Prop. 11.1 and preserves (binary) brackets, as shown in the proof of Thm. 11.2. Therefore there is a strict  $L_\infty$ -morphism from  $L_\infty(M, \omega)$  to  $L_\infty(LM, \omega^\ell)$ , whose only non-vanishing component is the map  $\alpha \mapsto \alpha^\ell$ . Hence, given a homotopy moment map for  $(M, \omega)$ , composing with the above  $L_\infty$ -morphism we get a homotopy moment map for  $(LM, \omega^\ell)$ . In [14] we extend Thm. 11.2 further.

## 12. RELATION TO OTHER WORKS

**12.1. Other notions of moment map.** Let  $M$  be a manifold endowed with a closed  $n+1$ -form  $\omega$ , and  $G$  a Lie group acting on  $M$  preserving  $\omega$ .

A **multimomentum map** in the sense of Cariñena-Crampin-Ibort [8, Sec. 4.2] is a map  $f_1: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^{n-1}(M)$  satisfying

$$-\iota_{v_x} \omega = d(f_1(x)) \quad \text{for all } x \in \mathfrak{g}.$$

Such maps are called **covariant momentum maps** in [16]; they are used there to study symmetries in classical field theories. Hence, if  $(f_k)$  is a homotopy moment map, then its first component is a multimomentum/covariant momentum map in the above sense.

If  $n = 2$  and  $x, y \in \mathfrak{g}$  are *commuting* elements, then the invariance of  $\omega$  under  $G$  implies that  $\iota(v_x \wedge v_y) \omega$  is a closed 1-form. Madsen-Swann [21, Sec. 2] define the **Lie kernel of  $\mathfrak{g}$**  as  $P = \ker([\cdot, \cdot]: \Lambda^2 \mathfrak{g} \rightarrow \mathfrak{g})$ . They define a **multi-moment map** as an equivariant map  $\nu: M \rightarrow P^*$  such that  $\iota(v_p) = \sum_i \iota(v_{x_i} \wedge v_{y_i}) \omega$  is exact with primitive  $\nu^* p$ , for any  $p = \sum_i x_i \wedge y_i \in P$ .

It is clear from Eq. (25), that if  $(f_1, f_2): \mathfrak{g} \rightarrow L_\infty(M, \omega)$  is an equivariant homotopy moment map, then the restriction of  $-f_2$  to the Lie kernel is a multi-moment map in the above sense.

**12.2. Actions on Courant algebroids.** Recall that a **Courant algebroid** consists of a vector bundle  $E \rightarrow M$  with a non-degenerate symmetric pairing on the fibers, a bilinear bracket  $\llbracket \cdot, \cdot \rrbracket$  on  $\Gamma(E)$ , and a bundle map  $\rho: E \rightarrow TM$  satisfying certain conditions, see for instance [27, Def. 4.2]. Courant algebroids appear naturally in the study of Dirac and generalized complex structures.

Let  $(M, \omega)$  be a pre-2-plectic manifold. There is a Courant algebroid associated to the closed 3-form  $\omega$ , namely  $TM \oplus T^*M$  with the natural pairing  $\langle X + \xi, Y + \eta \rangle = \xi(Y) + \eta(X)$ , the projection  $\rho$  onto the first factor, and bracket

$$\llbracket X + \xi, Y + \eta \rrbracket_\omega = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi + \iota_Y \iota_X \omega.$$

This Courant algebroid ~~by~~ is sometimes called the  $\omega$ -twisted Courant algebroid, and we will denote it by  $(TM \oplus T^*M)_\omega$ .

Bursztyn-Cavalcanti-Gualtieri [6] defined the notion of extended action on a Courant algebroid. We spell out only the case of a **trivially extended action** [6, Def. 2.12] on the above Courant algebroid: it is an action of a connected Lie group  $G$  on  $M$  together with a linear map  $\xi: \mathfrak{g} \rightarrow \Gamma(T^*M)$  such that  $\mathfrak{g} \rightarrow \Gamma(TM \oplus T^*M)_\omega, x \mapsto v_x + \xi(x)$  is bracket-preserving and the Lie algebra morphism  $x \mapsto \text{ad}_{v_x + \xi(x)} = \llbracket v_x + \xi(x), \cdot \rrbracket_\omega$  into the infinitesimal automorphisms of the Courant algebroid integrates to an action of  $G$  on  $TM \oplus T^*M$ .

The following lemma is essentially contained in [6, Sec. 2.2].

**Lemma 12.1.** *Let  $G$  act on pre-2-plectic manifold  $(M, \omega)$  and  $\mu \in (\mathfrak{g}^* \otimes \Omega^1(M))^G$ .  $\omega - \mu$  is an equivariant extension of  $\omega$  if and only if*

$$\Psi: \mathfrak{g} \rightarrow \Gamma(TM \oplus T^*M)_\omega, \quad x \mapsto v_x - \mu(x)$$

*is a trivially extended action on the Courant algebroid  $(TM \oplus T^*M)_\omega$  integrating to the action of  $G$  by tangent-cotangent lifts with isotropic image:*

$$\langle \text{im } \Psi, \text{im } \Psi \rangle = 0.$$

*Proof.* To simplify the notation, denote  $\xi = -\mu$ . Suppose  $\omega + \xi$  is an equivariant extension (so Eq. (17) holds). The following two statements are contained in the text after Prop. 2.11 of [6]. First, the condition  $\iota_{v_x} \omega - d(\xi(x)) = 0$  and the equivariance of  $\xi$  are equivalent to the fact that  $\Psi$  preserves brackets and  $TM \oplus \{0\}$  is a  $\mathfrak{g}$ -equivariant splitting. Second,  $\iota_{v_x} \omega - d(\xi(x)) = 0$  is also equivalent to  $\text{ad}_{v_x + \xi(x)}$  being the Lie derivative  $\mathcal{L}_{v_x}$ , acting on vector fields and 1-forms (hence  $\Psi$  integrates to the action of  $G$  on  $TM \oplus T^*M$  is by tangent-cotangent lifts). The condition  $\iota_{v_x}(\xi(x)) = 0$  clearly means that the image of  $\Psi$  is isotropic.

The converse implication is proven by reversing the above argument.  $\square$

On the other hand, we know that an equivariant extension  $\omega - \mu$  delivers a moment map  $(f_k): \mathfrak{g} \rightarrow L_\infty(M, \omega)$ , by Thm. 6.3. When  $\omega$  is non-degenerate, the relation between  $f$  and the extended action  $\Psi$  of Lemma 12.1 is simply

$$(45) \quad \Psi = i \circ f.$$

Here

$$(i_k): L_\infty(M, \omega) \rightarrow L_\infty((TM \oplus T^*M)_\omega)$$

is the embedding of Lie 2-algebras given in [24, Thm. 5.2]<sup>2</sup> (it is defined only when  $\omega$  is non-degenerate).  $L_\infty((TM \oplus T^*M)_\omega)$  denotes the Lie 2-algebra associated to the Courant algebroid  $(TM \oplus T^*M)_\omega$  by Weinstein-Roytenberg [28][26, Thm. 6.5]; its underlying graded vector space is  $C^\infty(M)$  in degree  $-1$  and  $\Gamma(TM \oplus T^*M)_\omega$  in degree zero, and the binary bracket in degree zero is the twisted Courant bracket, i.e. the skew-symmetrization of  $\llbracket \cdot, \cdot \rrbracket_\omega$ . In summary, the composition of the two Lie-2 algebra morphisms on the r.h.s. of Eq. (45) happens to be a strict morphism from  $\mathfrak{g}$  to  $L_\infty((TM \oplus T^*M)_\omega)$ , whose only non-trivial component is  $\Psi$ .

<sup>2</sup>The same theorem appears also as [26, Thm. 7.1] but with different sign conventions.

**Remark 12.2.** There is a notion of moment map for extended actions on Courant algebroids [6, Def. 2.14]. In the case of a trivially extended action, however, the only such moment map is the zero map.

Next we consider arbitrary homotopy moment maps:

**Proposition 12.3.** *Let  $(f_k): \mathfrak{g} \rightarrow L_\infty(M, \omega)$  be a moment map, not necessary arising from an equivariant extension of  $\omega$ , with unary component  $f_1: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^1(M)$ . Then*

$$\mathfrak{g} \rightarrow \text{aut}((TM \oplus T^*M)_\omega), x \mapsto \text{ad}_{v_x - f_1(x)}$$

*is a Lie algebra morphism into the infinitesimal automorphisms of the Courant algebroid.*

In other words, we obtain an infinitesimal action of  $\mathfrak{g}$  on the Courant algebroid, which however does not record all of the information of the moment map since the binary component  $f_2$  is lost.

The above can be checked by a straightforward computation, using Eq. (26). In the case that  $\omega$  is non-degenerate, a conceptual explanation is as follows.  $i \circ f$  is a Lie 2-algebra morphism (being a composition of such), so its unary component

$$\mathfrak{g} \rightarrow \Gamma(TM \oplus T^*M)_\omega, \quad x \mapsto v_x - f_1(x)$$

matches the Lie bracket on  $\mathfrak{g}$  and the twisted Courant bracket up to an exact 1-form. Now recall that the twisted Courant bracket and  $[\![\cdot, \cdot]\!]_\omega$ , applied to any two sections, also differ by an exact 1-form, and that exact 1-forms lie in the kernel of  $\text{ad}$ .

**12.3. Actions on differential graded manifolds.** Let  $M$  be a manifold endowed with a closed  $n+1$ -form  $\omega$ . Uribe [32] considers the graded manifold  $P = T[1]M \oplus \mathbb{R}[n]$ , whose graded algebra of function is  $\Omega(M) \otimes S[t]$ , where  $S[t]$  denotes polynomials in a variable  $t$  of degree  $n$ . The graded manifold  $P$  is endowed with the homological vector field  $Q = d_{\text{dR}} + \omega \partial_t$ , where  $d_{\text{dR}}$  is the de Rham vector field on  $T[1]M$ . Assume that a connected Lie group  $G$  acts on  $M$ , and assume for simplicity that  $\omega$  is preserved by the action.

Denote

$$\mathfrak{sym}(P, Q) = \mathfrak{X}_{<0}(P) \oplus \{Y \in \mathfrak{X}_0(P) : [Q, Y] = 0\}$$

(the vector fields of negative degree on  $P$ , together with the degree zero vector fields commuting with  $Q$ ). It is a DGLA with the usual bracket of vector fields and differential  $[Q, \bullet]$ . The main motivation for Uribe to study these objects is that, when  $n = 2$ ,  $\mathfrak{sym}(P, Q)$  is isomorphic to the DGLA of symmetries of the  $\omega$ -twisted exact Courant algebroid over  $M$ .  $\mathfrak{sym}(P, Q)$  contains a sub-DGLA  $\mathfrak{gsym}(P, Q)$ , whose degree zero component consists of vector fields preserving the function  $\omega$ , and with degree  $-1$  component

$$\{\iota_X + \alpha \partial_t : X \in \mathfrak{X}(M), \alpha \in \Omega^{n-1}(M), d\alpha = -\iota_X \omega\}$$

(see [32, Sec. 3.2]). Further, there is a DGLA associated to the Lie algebra  $\mathfrak{g}$  of  $G$ : it is<sup>3</sup>  $\mathfrak{g}[1] \oplus \mathfrak{g}$ , with bracket given by the bracket and adjoint action of  $\mathfrak{g}$ , and differential  $\text{id}_{\mathfrak{g}}$ .

We are now ready to reproduce two statements. First, [32, Lemma 3.7] states that strict morphisms of DGLAs

$$(46) \quad \mathfrak{g}[1] \oplus \mathfrak{g} \rightarrow \mathfrak{gsym}(P, Q)$$

lifting the action of  $\mathfrak{g}$  on  $M$  are in bijective correspondence with cocycles  $\omega - \mu$  in the Cartan model, where  $\mu \in \Omega^{n-1}(M) \otimes S^1 \mathfrak{g}^*[-2]$ .

Second, by [32, Prop. 2.15],  $L_\infty$ -morphisms

$$(47) \quad \text{from } \mathfrak{g}[1] \oplus \mathfrak{g} \text{ to } \mathfrak{sym}(P, Q)$$

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<sup>3</sup>It is a DGLA concentrated in degrees  $-1$  and  $0$ , which corresponds to the natural structure of Lie algebra crossed module of  $\mathfrak{g}$  over itself.

lifting the map  $\mathfrak{g}[1] \oplus \mathfrak{g} \rightarrow \mathfrak{X}(M)[1] \oplus \mathfrak{X}(M)$  induced by the action of  $\mathfrak{g}$  on  $M$  are in bijective correspondence with closed extensions of  $\omega$  in the BRST model  $\wedge \mathfrak{g}^* \otimes S\mathfrak{g}^* \otimes \Omega(M)$ .

Notice that, as  $\mathfrak{g}\mathfrak{sym}(P, Q)$  is a sub-DGLA of  $\mathfrak{sym}(P, Q)$ , the morphisms appearing in (46) are particular cases of those appearing in (47).

The relevance of the above to our work is as follows. A cocycle  $\omega - \mu$  in the Cartan model induces two kinds of objects: by Thm. 6.3 it induces a moment map of a particular form (equivariant, and determined by its unary component); by [32, Lemma 3.7] it induces an infinitesimal action of  $\mathfrak{g}[1] \oplus \mathfrak{g}$  on  $(P, Q)$  of a particular form (strict, and preserving  $\omega$ ). Proposition 2.15 in [32] suggests that there is a relation between arbitrary homotopy moment maps and closed extensions of  $\omega$  in the BRST model. We are pursuing such ideas in future work.

### 13. CONCLUDING REMARKS

This work raises many questions and suggests a variety of possible directions for future research. We mentioned some of these throughout the text. Here we give a few more.

First, several of the examples introduced in this paper deserve more thorough investigation. In particular:

- In Sec. 8.1, we consider moment maps arising from exact  $n$ -plectic forms. Manifolds equipped such  $n$ -plectic structures naturally arise in certain models of classical field theory [16]. How do homotopy moment maps relate to the conservation laws and symmetries studied within these models?
- In Sec. 8.3, we note a relationship between the homotopy moment map arising from a Lie group acting on itself via conjugation and the theory of quasi-Hamiltonian  $G$ -spaces. Moreover, if  $G$  acts on a pre-2-plectic manifold  $(M, \omega)$ , with  $\omega$  both  $G$ -invariant and representing a degree 3 integral cohomology class, then we also have reasons to suspect that there is a relationship between homotopy moment maps lifting the  $G$  action, and  $G$  equivariant  $U(1)$ -gerbes on  $M$ . Indeed, combining the results in Sec. 7 of [26] with those in Sec. 12 of [9] implies that a homotopy moment map lifts the  $\mathfrak{g}$  action on  $(M, \omega)$  to a  $\mathfrak{g}$  action on any  $U(1)$ -gerbe whose 3-curvature is  $\omega$ . Some relationships have already been established between trivializations of  $G$ -equivariant gerbes and quasi-Hamiltonian  $G$ -spaces (e.g. [15]). What is the precise relationship which intertwines these formalisms with homotopy moment maps?

Further development of the general theory of homotopy moment maps would also be desirable. For example:

- From a closed invariant form  $\omega$  on a  $G$ -manifold  $M$  (with  $G$  compact and connected), we explicitly constructed in Thm. 6.3 a moment map from a particular type of equivariant cocycle in the Cartan model. Specifically, the cocycle must be of the form  $\omega - \mu^{(1)}$ , with  $\mu^{(1)} \in (\Omega^{n-1}(M) \otimes \mathfrak{g}^*[-2])^G$ . We can also construct explicit maps from more general cocycles, such as those of the form  $\omega - \mu^{(1)} - \mu^{(2)}$ , where  $\mu^{(2)} \in (\Omega^{n-3}(M) \otimes S^2(\mathfrak{g}^*[-2]))^G$ . Directly verifying that such a cocycle gives a moment map requires copious amounts of bookkeeping, so this result was not included in the text. But based on this and other evidence, we believe that any equivariant extension  $\omega$  corresponds to a homotopy moment map. We are currently developing a more elegant approach [14] to the general theory in order to verify this conjecture, and we believe that it will provide a more satisfactory understanding of the relationship between equivariant cohomology and homotopy moment maps. Perhaps it is not surprising that in this new approach, we rely on more conceptual models for equivariant cohomology, instead of using the Cartan complex.

- When should two moment maps be considered equivalent? Indeed, there are well-known uniqueness results for moment maps in symplectic geometry and, for example, abstract moment maps in Hamiltonian cobordism theory. For us, the question is particularly relevant since some of our constructions (e.g. Thm. 9.7) depend on a number of choices. We briefly discussed uniqueness issues in Sec. 7.4 for the case of pre-2-plectic manifolds. We also mentioned earlier in Remark 3.7 that there is already a notion of a homotopy between  $L_\infty$ -morphisms available via the theory of Quillen model categories. Work in progress suggests that a variant of this notion of homotopy will likely be the correct one.
- Can one perform reduction of pre- $n$ -plectic forms using an equivariant moment map  $(f_k): \mathfrak{g} \rightarrow L_\infty(M, \omega)$ , analogously to the Marsden-Weinstein-Meyer reduction in symplectic geometry? It is easily checked that, if

$$S = \{p \in M : f_1(x)|_p = 0 \text{ for all } x \in \mathfrak{g}\}$$

satisfies certain regularity conditions, then it is  $G$ -invariant and the pullback of  $\omega$  descends to  $S/G$ . However  $S$  is the vanishing set of a family of  $(n-1)$ -forms, and it is hard to control its smoothness properties. The only instance we are aware of where the above-mentioned reduction procedure works, apart from the one discussed in Subsection 10.5, is Ex. 8.2, namely the (free and proper) cotangent lift action of  $G$  on  $\Lambda^n T^*N$ : in this case  $S/G$  is canonically isomorphic to  $\Lambda^n T^*(N/G)$  with its canonical  $n$ -plectic form. This procedure is probably too naïve in general, and it only uses a small part of the information provided by the moment map.

#### APPENDIX A. EXPLICIT FORMULAS FOR $L_\infty$ -MORPHISMS

Here we recall the relationship between  $L_\infty$ -morphisms and morphisms of dg coalgebras. We use this to prove Prop. 3.8 from Section 3 and other results needed throughout the text. These are well-known to experts, but it is difficult to find explicit formulas in the literature. Our exposition is self-contained for the reader's convenience. For more details on coalgebras, we suggest Sections 3d and 22a of [11], and Appendix B of [23]. The description of  $L_\infty$ -algebras as coalgebras is also reviewed in Section 2 of [20].

Throughout this section, graded means  $\mathbb{Z}$ -graded and all vector spaces are over  $\mathbb{R}$ . If  $V$  is a graded vector space, then  $\mathbf{s}V$  (resp.  $\mathbf{s}^{-1}V$ ) denotes the **suspension** (resp. **desuspension**) of  $V$  i.e.

$$(\mathbf{s}V)_i = V_{i-1}, \quad (\mathbf{s}^{-1}V)_i = V_{i+1}.$$

Unlike the aforementioned references, we adopt cohomological conventions i.e. our (co)differentials have degree  $+1$ .

**A.1. Coalgebras.** A coalgebra  $(C, \Delta)$  is a graded vector space  $C$  equipped with a linear map  $\Delta: C \rightarrow C \otimes C$  called the **comultiplication** such that

$$\Delta(C_i) \subset \bigoplus_{j+k=i} C_j \otimes C_k,$$

and

$$(\Delta \otimes \text{id}) \Delta = (\text{id} \otimes \Delta) \Delta.$$

The above condition is called **coassociativity**. If  $(C, \Delta)$  is a coalgebra, denote by  $T: C \otimes C \rightarrow C$  the twist map  $T(x \otimes y) = (-1)^{|x||y|} y \otimes x$ . We say  $(C, \Delta)$  is **cocommutative** iff  $T \circ \Delta = \Delta$ . A **counit** for  $(C, \Delta)$  is a linear map  $\epsilon: C \rightarrow \mathbb{R}$  such that  $(\epsilon \otimes \text{id}) \Delta = (\text{id} \otimes \epsilon) \Delta = \text{id}$ . Such a coalgebra is **coaugmented** iff it is equipped with an injective linear map

$$\begin{aligned} \mathbb{R} &\hookrightarrow C \\ 1 &\mapsto 1_C \end{aligned}$$

such that  $\epsilon(1_C) = 1$  and  $\Delta(1_C) = 1_C \otimes 1_C$ . We write  $\bar{C} = \ker \epsilon$  so that  $C \cong \mathbb{R} \oplus \bar{C}$  as vector spaces. Given a coaugmented coalgebra, we can define the **reduced comultiplication**  $\bar{\Delta}: \bar{C} \rightarrow \bar{C} \otimes \bar{C}$  by

$$\bar{\Delta}c = \Delta c - c \otimes 1 - 1 \otimes c.$$

This makes  $\bar{C}$  a coalgebra without a counit.

Denote the **reduced diagonal** by  $\bar{\Delta}^{(n)}$ . It is recursively defined by the formulas:

$$\begin{aligned} \bar{\Delta}^{(0)} &= \text{id} \\ \bar{\Delta}^{(1)} &= \bar{\Delta} \\ \bar{\Delta}^{(n)} &= (\bar{\Delta} \otimes \text{id}^{\otimes(n-1)}) \circ \bar{\Delta}^{(n-1)}: \bar{C} \rightarrow \bar{C}^{\otimes(n+1)}. \end{aligned}$$

A simple induction argument shows that we can rewrite  $\bar{\Delta}^{(n)}$  as

$$(48) \quad \bar{\Delta}^{(n)} = (\bar{\Delta}^{(n-1)} \otimes \text{id}) \circ \bar{\Delta}.$$

Every coaugmented coalgebra  $(C, \Delta, \epsilon, 1_C)$  has a canonical filtration defined recursively by the formulas

$$\begin{aligned} F_0 C &= \mathbb{R} \cdot 1_C \\ F_k C &= \{x \in \bar{C} \mid \bar{\Delta}x \in F_{k-1}C \otimes F_{k-1}C\}. \end{aligned}$$

Such a coalgebra is **connected** iff

$$C = \bigcup F_k C.$$

If  $(C, \Delta, \epsilon, 1_C)$  is connected, then the coaugmentation  $1_C$  is unique (Prop. 3.1 Sec. B3 [23]).

A.1.1. *The coalgebra  $S(V)$ .* Given a graded vector space  $V$ , the graded symmetric algebra

$$\begin{aligned} S(V) &= \mathbb{R} \oplus V \oplus S^2(V) \oplus S^3(V) \oplus \dots \\ &= \mathbb{R} \oplus \bar{S}(V) \end{aligned}$$

is naturally a coaugmented cocommutative coalgebra. The comultiplication  $\Delta$  is the unique morphism of algebras such that  $\Delta(v) = v \otimes 1 + 1 \otimes v$  for all  $v \in V$ . The counit is the projection  $S(V) \rightarrow \mathbb{R}$ , and the coaugmentation is the inclusion  $\mathbb{R} \hookrightarrow S(V)$ . The reduced comultiplication  $\bar{\Delta}$  on  $\bar{S}(V)$  is explicitly:

$$\begin{aligned} \bar{\Delta}(v_1 \odot v_2 \odot \dots \odot v_n) &= \sum_{1 \leq p \leq n-1} \sum_{\sigma \in \text{Sh}(p, n-p)} \epsilon(\sigma) (v_{\sigma(1)} \odot v_{\sigma(2)} \odot \dots \odot v_{\sigma(p)}) \\ &\quad \otimes (v_{\sigma(p+1)} \odot v_{\sigma(p+2)} \odot \dots \odot v_{\sigma(n)}). \end{aligned}$$

Using Eq. (48), it is easy to see that

$$\bar{S}^{\bullet \leq k}(V) \subseteq \ker \bar{\Delta}^{(k)}.$$

The following result will be useful in the proceeding sections.

**Lemma A.1.** *If  $v_1 \odot v_2 \odot \dots \odot v_n \in \bar{S}(V)$ , and  $1 \leq p \leq n-1$  then*

$$\begin{aligned} \bar{\Delta}^{(p)}(v_1 \odot \dots \odot v_n) &= \sum_{\substack{k_1+k_2+\dots+k_{p+1}=n \\ k_1, k_2, \dots, k_{p+1} \geq 1}} \sum_{\sigma \in \text{Sh}(k_1, k_2, \dots, k_{p+1})} \epsilon(\sigma) v_{\sigma(1)} \odot \dots \odot v_{\sigma(k_1)} \\ &\quad \otimes v_{\sigma(k_1+1)} \odot \dots \odot v_{\sigma(k_1+k_2)} \otimes v_{\sigma(k_1+k_2+1)} \odot \dots \odot v_{\sigma(k_1+k_2+k_3)} \otimes \dots \\ &\quad \otimes v_{\sigma(m-k_{p+1}+1)} \odot \dots \odot v_{\sigma(n)}. \end{aligned}$$

In particular, we have

$$\bar{\Delta}^{(n-1)}(v_1 \odot \dots \odot v_n) = \sum_{\sigma \in \mathcal{S}_n} \epsilon(\sigma) v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)}.$$



*Proof.* It is clear from the definition of  $\bar{\Delta}$  that the statement holds for  $p = 1$ . Proceeding by induction, assume it also holds for  $p - 1$ . By Eq. (48), we have

$$\begin{aligned}\bar{\Delta}^{(p)}(v_1 \odot \cdots \odot v_n) &= (\bar{\Delta}^{(p-1)} \otimes \text{id}) \circ \bar{\Delta}(v_1 \odot \cdots \odot v_n) \\ &= (\bar{\Delta}^{(p-1)} \otimes \text{id}) \left( \sum_{1 \leq j \leq n-1} \sum_{\sigma \in \text{Sh}(j, n-j)} \epsilon(\sigma) (v_{\sigma(1)} \odot v_{\sigma(2)} \odot \cdots \odot v_{\sigma(j)}) \right. \\ &\quad \left. \otimes (v_{\sigma(j+1)} \odot v_{\sigma(j+2)} \odot \cdots \odot v_{\sigma(n)}) \right)\end{aligned}$$

Since  $\bar{S}^{\bullet \leq p-1}(V) \subseteq \ker \bar{\Delta}^{(p-1)}$ , the last line above gives

$$(49) \quad \begin{aligned}\bar{\Delta}^{(p)}(v_1 \odot \cdots \odot v_n) &= \sum_{j=p}^{n-1} \sum_{\sigma \in \text{Sh}(j, n-j)} \epsilon(\sigma) \bar{\Delta}^{(p-1)}(v_{\sigma(1)} \odot v_{\sigma(2)} \odot \cdots \odot v_{\sigma(j)}) \\ &\quad \otimes (v_{\sigma(j+1)} \odot v_{\sigma(j+2)} \odot \cdots \odot v_{\sigma(n)}).\end{aligned}$$

The induction hypothesis implies

$$\begin{aligned}\bar{\Delta}^{(p-1)}(v_{\sigma(1)} \odot v_{\sigma(2)} \odot \cdots \odot v_{\sigma(j)}) &= \sum_{k_1, k_2, \dots, k_p \geq 1}^{k_1 + k_2 + \cdots + k_p = j} \sum_{\tau \in \text{Sh}(k_1, k_2, \dots, k_p)} \epsilon(\tau) v_{\sigma'(1)} \odot \cdots \odot v_{\sigma'(k_1)} \\ &\quad \otimes v_{\sigma'(k_1+1)} \odot \cdots \odot v_{\sigma'(k_1+k_2)} \otimes \cdots \otimes v_{\sigma'(j-k_p+1)} \odot \cdots \odot v_{\sigma'(j)},\end{aligned}$$

where  $\sigma'$  is the permutation  $\tau \circ \sigma$ . Substituting this expression into Eq. (49) and setting  $k_{p+1} = n - j$  gives the desired result. Note that for  $p = n - 1$ , we have  $k_1 = k_2 = \cdots k_{p+1} = 1$  and therefore  $\text{Sh}(k_1, k_2, \dots, k_{p+1}) = \mathcal{S}_n$ .  $\square$

The above Lemma implies that  $\ker \bar{\Delta}^{(k)} = \bar{S}^{\bullet \leq k}(V)$  for  $k \geq 0$  and

$$(50) \quad \bar{S}(V) = \bigcup_n \ker \bar{\Delta}^{(n)}.$$

For  $n \geq 1$ , the canonical filtration  $F_n S(V)$  corresponds to the filtration on  $\bar{S}(V)$  given by  $\ker \bar{\Delta}^{(n)}$ . Hence,  $S(V)$  is a connected coalgebra.

**A.2. Coalgebra morphisms.** A morphism between connected coalgebras  $(C, \Delta, \epsilon, 1_C)$  and  $(C, \Delta', \epsilon', 1_{C'})$  is a degree 0 linear map  $F: C \rightarrow C'$  such that

$$\begin{aligned}\Delta' \circ F &= (F \otimes F) \circ \Delta \\ \epsilon &= \epsilon' \circ F.\end{aligned}$$

Since  $\mathbb{R}$  is a field, it follows from the above identities that  $F$  automatically preserves the coaugmentations, and therefore is uniquely determined by its restriction to  $\bar{C}$ . Hence, morphisms between such coalgebras correspond to degree 0 linear maps  $F: \bar{C} \rightarrow \bar{C}'$  satisfying  $\bar{\Delta}' \circ F = (F \otimes F) \circ \bar{\Delta}$ .

The coalgebra  $S(V)$  is **cofree** in the category of connected cocommutative coalgebras. (See Prop. 4.1 in Sec. B3 of [23] or Lemma 22.1 in [11]). Namely, if  $C = \mathbb{R} \oplus \bar{C}$  is a connected cocommutative coalgebra and  $f: \bar{C} \rightarrow V$  is a degree 0 linear map to a graded vector space  $V$ , then there exists a unique morphism of connected coalgebras  $F: C \rightarrow S(V)$  such that  $\text{pr}_V \circ F|_{\bar{C}} = f$ , where  $\text{pr}_V: S(V) \rightarrow V$  is the projection.

We consider this universal property in more detail for the case when  $C$  is also a symmetric algebra. In order to make contact with the theory of  $L_\infty$ -algebras, we work with the desuspension  $\mathbf{s}^{-1}V$  of  $V$ . Consider a degree zero linear map

$$F^1: \bar{S}(\mathbf{s}^{-1}V) \rightarrow \mathbf{s}^{-1}V',$$



where  $V'$  is a graded vector space. We denote by  $F_k^1$  the restrictions  $F^1|_{\bar{S}^k(\mathfrak{s}^{-1}V)}$  so that  $F^1 = F_1^1 + F_2^1 + \dots$ . There is a one-to-one correspondence between such maps  $F^1$  and collections of graded skew-symmetric linear maps

$$(51) \quad f_k: V^{\otimes k} \rightarrow V', \quad |f_k| = 1 - k \quad k \geq 1.$$

given by

$$(52) \quad F_k^1 = (-1)^{\frac{k(k-1)}{2}} \mathfrak{s}^{-1} \circ f_k \circ \mathfrak{s}^{\otimes k}.$$

**Proposition A.2.** *If  $V$  and  $V'$  are graded vector spaces, and  $F^1: \bar{S}(\mathfrak{s}^{-1}V) \rightarrow \mathfrak{s}^{-1}V'$  is a degree zero linear map, then there exists a unique morphism of coalgebras*

$$F: S(\mathfrak{s}^{-1}V) \rightarrow S(\mathfrak{s}^{-1}V')$$

*lifting  $F^1$  such that  $\text{pr}_{\mathfrak{s}^{-1}V'} \circ F|_{\bar{S}(\mathfrak{s}^{-1}V)} = F^1$ , where  $\text{pr}_{\mathfrak{s}^{-1}V'}$  is the projection to  $\mathfrak{s}^{-1}V'$ .*

*Proof.* Again, the statement is nothing but a special case of the aforementioned universal property. Here we just recall the construction of the coalgebra morphism  $F$ . First, define for each  $p > 0$ :

$$\psi^{(p)}: \underbrace{\bar{S}(\mathfrak{s}^{-1}V) \otimes \bar{S}(\mathfrak{s}^{-1}V) \otimes \dots \otimes \bar{S}(\mathfrak{s}^{-1}V)}_p \rightarrow \bar{S}^p(\mathfrak{s}^{-1}V')$$

$$\psi^{(p)}(c_1 \otimes c_2 \otimes \dots \otimes c_p) = \frac{1}{p!} F_{k_1}^1(c_1) \odot \dots \odot F_{k_p}^1(c_p),$$

where  $c_1, \dots, c_p$  are simple tensors with  $c_i \in \bar{S}^{k_i}(\mathfrak{s}^{-1}V)$ . Define  $F$  to be:

$$F(1) = 1$$

and

$$(53) \quad F(c) = \sum_{p=0}^{\infty} \psi^{(p+1)} \circ \bar{\Delta}^{(p)}(c), \quad c \in \bar{S}(\mathfrak{s}^{-1}V).$$

Note the infinite sum is well-defined since Eq. (50) holds.  $\square$

We use Lemma A.1 to write out the formula for  $F$  explicitly in terms of the maps  $F_k^1$ . Given  $v_1, \dots, v_n \in \mathfrak{s}^{-1}V$ , Eq. (53) implies that

$$F(v_1 \odot \dots \odot v_n) = \sum_{p=0}^{n-1} \psi^{(p+1)} \circ \bar{\Delta}^{(p)}(v_1 \odot \dots \odot v_n)$$

since  $\ker \bar{\Delta}^{(p)} = \bar{S}^{\bullet \leq p}(\mathfrak{s}^{-1}V)$ . Therefore

$$(54) \quad \begin{aligned} F(v_1 \odot \dots \odot v_n) &= F_n^1(v_1 \odot \dots \odot v_n) + \sum_{p=1}^{n-1} \sum_{k_1+k_2+\dots+k_{p+1}=n} \sum_{\sigma \in \text{Sh}(k_1, k_2, \dots, k_{p+1})} \frac{\epsilon(\sigma)}{(p+1)!} \\ &\quad \times F_{k_1}^1(v_{\sigma(1)} \odot \dots \odot v_{\sigma(k_1)}) \odot F_{k_2}^1(v_{\sigma(k_1+1)} \odot \dots \odot v_{\sigma(k_1+k_2)}) \odot \dots \\ &\quad \odot F_{k_{p+1}}^1(v_{\sigma(m-k_{p+1}+1)} \odot \dots \odot v_{\sigma(n)}). \end{aligned}$$

We also define projections  $F_n^p$  for  $p > 1$  by

$$(55) \quad F_n^p = \text{pr}_{\bar{S}^p(\mathfrak{s}^{-1}L')} \circ F|_{\bar{S}^n(\mathfrak{s}^{-1}L')}: \bar{S}^n(\mathfrak{s}^{-1}L) \rightarrow \bar{S}^p(\mathfrak{s}^{-1}L'),$$

and then Eq. (54) implies that

$$(56) \quad \begin{aligned} F_n^p(v_1 \odot \dots \odot v_n) &= \sum_{k_1+k_2+\dots+k_p=n} \sum_{\sigma \in \text{Sh}(k_1, k_2, \dots, k_p)} \frac{\epsilon(\sigma)}{p!} F_{k_1}^1(v_{\sigma(1)} \odot \dots \odot v_{\sigma(k_1)}) \\ &\quad \odot F_{k_2}^1(v_{\sigma(k_1+1)} \odot \dots \odot v_{\sigma(k_1+k_2)}) \odot \dots \odot F_{k_p}^1(v_{\sigma(m-k_p+1)} \odot \dots \odot v_{\sigma(n)}). \end{aligned}$$

In particular,

$$F_n^n(v_1 \odot \cdots \odot v_n) = F_1^1(v_1) \odot F_1^1(v_2) \odot \cdots \odot F_1^1(v_n),$$

and

$$F_n^p(v_1 \odot \cdots \odot v_n) = 0 \quad \text{for } p > n.$$

**A.3.  $L_\infty$ -algebras as dg coalgebras.** A **codifferential** of degree 1 on a connected coalgebra  $(C, \Delta, \epsilon, 1_C)$  is a linear map  $Q: C^i \rightarrow C^{i+1}$  satisfying the identities

$$Q(1_C) = 0, \quad Q \circ Q = 0,$$

and the coLeibniz identity

$$\Delta Q = (Q \otimes \text{id}) \Delta + (\text{id} \otimes Q) \Delta.$$

Such a codifferential on  $C$  is uniquely determined by its restriction to  $\bar{C}$  which satisfies the coLeibniz identity with respect to  $\bar{\Delta}$ .

**Theorem A.3** (Thm. 2.3 [20]). *An  $L_\infty$ -structure  $(l_k)$  on a graded vector space  $L$  uniquely determines a degree 1 codifferential  $Q$  on the coalgebra*

$$C(L) = S(\mathbf{s}^{-1}L).$$

*Conversely, any such codifferential on  $C(L)$  uniquely determines an  $L_\infty$ -structure on  $L$ .*

We briefly describe this correspondence given by the theorem. Consider the restrictions

$$Q_k = Q|_{\bar{S}^k(\mathbf{s}^{-1}L)}: \bar{S}^k(\mathbf{s}^{-1}L) \rightarrow \bar{S}(\mathbf{s}^{-1}L), \quad 1 \leq k < \infty$$

so that  $Q = Q_1 + Q_2 + Q_3 + \dots$ , and the projections

$$(57) \quad Q_m^k = \text{pr}_{\bar{S}^k(\mathbf{s}^{-1}L)} \circ Q_m: \bar{S}^m(\mathbf{s}^{-1}L) \rightarrow \bar{S}^k(\mathbf{s}^{-1}L).$$

In particular, it follows from Lemma 2.4 in [20] that  $Q$  is uniquely determined by the collection of maps

$$Q_k^1 = \text{pr}_{\mathbf{s}^{-1}L} \circ Q_k: \bar{S}^k(\mathbf{s}^{-1}L) \rightarrow \mathbf{s}^{-1}L, \quad k \geq 1.$$

These are related to the skew-symmetric “structure maps”  $l_k: L^{\otimes k} \rightarrow L$  via the formula

$$(58) \quad Q_k^1 = (-1)^{\frac{k(k-1)}{2}} \mathbf{s}^{-1} \circ l_k \circ \mathbf{s}^{\otimes k},$$

while the entire coderivation  $Q$  can be expressed as

$$(59) \quad Q_m(\mathbf{s}^{-1}x_1 \odot \cdots \odot \mathbf{s}^{-1}x_m) = Q_m^1(\mathbf{s}^{-1}x_1 \odot \cdots \odot \mathbf{s}^{-1}x_m) + \sum_{i=1}^{m-1} \sum_{\sigma \in \text{Sh}(i, m-i)} \epsilon(\sigma) Q_i^1(\mathbf{s}^{-1}x_{\sigma(1)} \odot \cdots \odot \mathbf{s}^{-1}x_{\sigma(i)}) \odot \mathbf{s}^{-1}x_{\sigma(i+1)} \odot \cdots \odot \mathbf{s}^{-1}x_{\sigma(m)},$$

for all  $x_i \in L$ . The condition  $Q \circ Q = 0$  is equivalent to the generalized Jacobi identity (6) for the collection  $(l_k)$ . In particular, it implies that  $l_1$  is degree +1 differential on  $L$ .

**A.4.  $L_\infty$ -Morphisms: General case.** A morphism  $F: (C, Q) \rightarrow (C', Q')$  between connected dg coalgebras is a coalgebra morphism such that

$$FQ = Q'F.$$

Thanks to Thm A.3, it is now clear what an  $L_\infty$ -morphism should be.

**Definition A.4.** A **morphism** between  $L_\infty$ -algebras  $(L, l_k)$  and  $(L', l'_k)$  is a morphism between the corresponding dg-coalgebras:

$$F: (C(L), Q) \rightarrow (C(L'), Q')$$

The following easy proposition says that ‘strict morphisms’ in the sense of Def. 3.3 are precisely those coalgebra morphisms that satisfy

$$\forall k \geq 2 \quad F_k^1 = 0.$$

**Proposition A.5.** *If  $(L, l_k)$  and  $(L', l'_k)$  are  $L_\infty$ -algebras, and  $f: L \rightarrow L'$  is a degree zero linear map satisfying*

$$l'_k \circ f^{\otimes k} = f \circ l_k \quad \forall k \geq 1,$$

*then the morphism  $F: (C(L), Q) \rightarrow (C(L'), Q')$  given by*

$$F(\mathbf{s}^{-1}x_1 \odot \cdots \mathbf{s}^{-1}x_k) = \mathbf{s}^{-1}f(x_1) \odot \cdots \odot \mathbf{s}^{-1}f(x_k)$$

*is a strict  $L_\infty$ -morphism.*

If  $F: (C(L), Q) \rightarrow (C(L'), Q')$  is an  $L_\infty$ -morphism, then the projections defined in Eq. (55) and Eq. (57) allow us to write the equality  $FQ = Q'F$  as

$$(60) \quad \sum_{k=1}^m F_k^1 Q_m^k = \sum_{k=1}^m Q_k'^1 F_m^k \quad \forall m \geq 1.$$

The results presented in Sec. A.2 imply that every such  $F$  is of the form (53), since by Prop. A.2 it is the unique lift of its projection  $F^1 = F_1^1 + F_2^1 + F_3^1 + \cdots$ . Hence,  $F$  can be uniquely expressed by its corresponding collection of **structure maps**  $(f_k)$  defined in (51). It is easy to see that the equality  $FQ = Q'F$  implies that the degree zero map

$$f_1: (L, l_1) \rightarrow (L', l'_1)$$

is a morphism of cochain complexes. This leads us to the notion of  $L_\infty$ -quasi-isomorphism given in Def. 3.6.

**A.5. Lie algebras.** Any differential graded Lie algebra (DGLA) can be thought of as a  $L_\infty$ -algebra via the ‘Chevalley-Eilenberg construction’, which associates to  $(\mathfrak{g}, d, [\cdot, \cdot])$  the coalgebra  $S(\mathbf{s}^{-1}\mathfrak{g})$  with codifferential  $D$  defined by the equations

$$(61) \quad \begin{aligned} D_1(\mathbf{s}^{-1}x) &= \mathbf{s}^{-1}dx \\ D_2(\mathbf{s}^{-1}x, \mathbf{s}^{-1}y) &= (-1)^{|x|}\mathbf{s}^{-1}[x, y] \\ D_k^1 &= 0, \quad k \geq 3. \end{aligned}$$

It’s easy to see that a DGLA morphism  $f: \mathfrak{g} \rightarrow \mathfrak{g}'$  induces a unique strict  $L_\infty$ -morphism between  $(S(\mathbf{s}^{-1}\mathfrak{g}), D)$  and  $(S(\mathbf{s}^{-1}\mathfrak{g}'), D')$ . Ordinary Lie algebras are those DGLA concentrated in degree zero with differential  $d = 0$ .

Now we consider  $L_\infty$ -algebra morphisms whose sources are just Lie algebras  $(\mathfrak{g}, [\cdot, \cdot])$ . Since the projections  $D_m^k$  are built from the structure maps  $D_m^1$  via Eq. (59), we have

$$D_m^k = 0 \quad \text{whenever } k \neq m - 1.$$

Therefore, Eq. (60), which a coalgebra morphism  $F: S(\mathbf{s}^{-1}\mathfrak{g}) \rightarrow S(\mathbf{s}^{-1}L)$  must satisfy to be an  $L_\infty$ -morphism, simplifies to

$$(62) \quad \begin{aligned} Q_1^1 F_1^1 &= 0, \\ F_{m-1}^1 D_m^{m-1} &= \sum_{k=1}^m Q_k^1 F_m^k \quad \forall m \geq 2. \end{aligned}$$

In particular, homotopy moment maps (Def. 5.1) are  $L_\infty$ -morphisms from a Lie algebra to a Lie  $n$ -algebra  $(L, l_k)$  satisfying Property (P), which we defined in Sec. 3.2 as being:

$$\forall k \geq 2 \quad l_k(x_1, \dots, x_k) = 0 \quad \text{whenever} \quad \sum_{i=1}^k |x_i| < 0.$$

Equation (58) implies that this is equivalent to the corresponding codifferential  $Q$  on  $\bar{S}(\mathbf{s}^{-1}L)$  satisfying

$$\forall k \geq 2 \quad Q_k^1(\mathbf{s}^{-1}x_1 \odot \dots \odot \mathbf{s}^{-1}x_k) = 0 \quad \text{whenever} \quad |\mathbf{s}^{-1}x_1 \odot \dots \odot \mathbf{s}^{-1}x_k| < k.$$

**Proposition A.6.** *If  $(\mathfrak{g}, [\cdot, \cdot])$  is a Lie algebra and  $(L, l_k)$  is a Lie  $n$ -algebra satisfying Property (P), then a coalgebra morphism  $F: \bar{S}(\mathbf{s}^{-1}\mathfrak{g}) \rightarrow \bar{S}(\mathbf{s}^{-1}L)$  is an  $L_\infty$ -algebra morphism if and only if*

$$(63) \quad F_{m-1}^1 D_m^{m-1} = Q_1^1 F_m^1 + Q_m^1 F_m^m$$

for  $2 \leq m \leq n$ , and

$$(64) \quad F_n^1 D_{n+1}^n = Q_{n+1}^1 F_{n+1}^{n+1},$$

where  $D$  and  $Q$  are the codifferentials determined by  $[\cdot, \cdot]$ , and  $(l_k)$ , respectively.

*Proof.* We will show the conditions given in Eqs. (63) and (64) are equivalent to those in (62). First, note that for any coalgebra morphism  $F: \bar{S}(\mathbf{s}^{-1}\mathfrak{g}) \rightarrow \bar{S}(\mathbf{s}^{-1}L)$

$$Q_1^1 F_1^1 = 0$$

holds trivially, since  $F_1^1$  is a degree 0 map and  $\mathbf{s}^{-1}\mathfrak{g}$  is in degree -1, while  $Q_1^1$  has degree +1 and  $\mathbf{s}^{-1}L$  is concentrated in degrees  $-n, \dots, -1$ . Next, we observe that Property (P) and Eq. (56) imply that

$$\sum_{k=1}^m Q_k^1 F_m^k = Q_1^1 F_m^1 + Q_m^1 F_m^m \quad \forall m \geq 2.$$

When  $m \geq n+1$ , the degree condition on  $\mathbf{s}^{-1}L$  implies that  $F_m^1 = 0$  and hence

$$\begin{aligned} Q_1^1 F_m^1 &= 0 \quad \forall m \geq n+1, \\ F_{m-1}^1 D_m^{m-1} &= 0 \quad \forall m \geq n+2. \end{aligned}$$

For the same reason,  $Q_m^1 = 0$  whenever  $m \geq n+2$ . Therefore

$$Q_m^1 F_m^m = 0 \quad \forall m \geq n+2.$$

Hence, satisfying Eqs. (63) and (64) is both necessary and sufficient for  $F$  to be an  $L_\infty$ -morphism.  $\square$

We now prove Prop. 3.8 as a corollary of the above.

**Corollary A.7** (Prop. 3.8). *If  $(\mathfrak{g}, [\cdot, \cdot])$  is a Lie algebra and  $(L, l_k)$  is a Lie  $n$ -algebra satisfying property (P), then a collection of  $n$  skew-symmetric maps*

$$f_m: \mathfrak{g}^{\otimes m} \rightarrow L, \quad |f_m| = 1 - m, \quad 1 \leq m \leq n$$

determine an  $L_\infty$ -morphism  $\bar{S}(\mathbf{s}^{-1}\mathfrak{g}) \rightarrow \bar{S}(\mathbf{s}^{-1}L)$  via Eq. (52) if and only if  $\forall x_i \in \mathfrak{g}$

$$(65) \quad \sum_{1 \leq i < j \leq m} (-1)^{i+j+1} f_{m-1}([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_m) \\ = l_1 f_m(x_1, \dots, x_m) + l_m(f_1(x_1), \dots, f_1(x_m)).$$

for  $2 \leq m \leq n$  and

$$(66) \quad \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+1} f_n([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{n+1}) = l_{n+1}(f_1(x_1), \dots, f_1(x_{n+1})).$$

*Proof.* Assume we are given such maps  $f_1, \dots, f_n$  satisfying the above equalities. Using Eq. (52), we construct the corresponding degree 0 maps  $F_1^1, \dots, F_n^1$ , and set  $F_k^1 = 0$  for  $k \geq n+1$ . By Prop. A.2, these give a unique coalgebra morphism  $F: \bar{S}(\mathfrak{s}^{-1}\mathfrak{g}) \rightarrow \bar{S}(\mathfrak{s}^{-1}L)$ . To show  $F$  is an  $L_\infty$ -morphism, Prop. A.6 implies it is sufficient to show Eqs. (63) and (64) hold. From Eq. (61), we have the equality  $D_2^1(\mathfrak{s}^{-1}x \odot \mathfrak{s}^{-1}y) = \mathfrak{s}^{-1}[x, y]$ , while Eq. (59) implies that

$$\begin{aligned} D_m^{m-1}(\mathfrak{s}^{-1}x_1 \odot \dots \odot \mathfrak{s}^{-1}x_m) &= \sum_{\sigma \in \text{Sh}(2, m-2)} \epsilon(\sigma) D_2^1(\mathfrak{s}^{-1}x_{\sigma(1)} \odot \mathfrak{s}^{-1}x_{\sigma(2)}) \odot \dots \odot \mathfrak{s}^{-1}x_{\sigma(m)} \\ &= \sum_{1 \leq i < j \leq m} (-1)^{i+j+1} \mathfrak{s}^{-1}[x_i, x_j] \odot \mathfrak{s}^{-1}x_1 \odot \dots \odot \widehat{\mathfrak{s}^{-1}x_i} \odot \dots \odot \widehat{\mathfrak{s}^{-1}x_j} \odot \dots \odot \mathfrak{s}^{-1}x_m. \end{aligned}$$

The signs in the last equality above are due to the fact that  $\mathfrak{g}$  is in degree 0. It follows from Eq. (52) that

$$(67) \quad F_m^1(\mathfrak{s}^{-1}x_1 \odot \dots \odot \mathfrak{s}^{-1}x_m) = \mathfrak{s}^{-1}f_m(x_1, \dots, x_m).$$

Therefore, the left-hand sides of Eqs. (63) and (64) are the desuspension of the left-hand sides of Eqs. (65) and (66), respectively.

Now we consider the right-hand sides. First, note that Eq. (67) also implies that

$$(68) \quad Q_1^1 F_m^1 = \mathfrak{s}^{-1}l_1 \circ f_m.$$

Recall Eq. (56) gives

$$F_m^m(\mathfrak{s}^{-1}x_1 \odot \dots \odot \mathfrak{s}^{-1}x_m) = F_1^1(\mathfrak{s}^{-1}x_1) \odot F_1^1(\mathfrak{s}^{-1}x_2) \odot \dots \odot F_1^1(\mathfrak{s}^{-1}x_m).$$

For each  $x_i$ , we have  $|F_1^1(\mathfrak{s}^{-1}x_i)| = -1$  and  $F_1^1(\mathfrak{s}^{-1}x_i) = \mathfrak{s}^{-1}f_1(x_i)$ . Therefore,

$$Q_1^1 F_m^m(\mathfrak{s}^{-1}x_1 \odot \dots \odot \mathfrak{s}^{-1}x_m) = \mathfrak{s}^{-1}l_m(f_1(x_1), \dots, f_1(x_m)).$$

Combining the above equality with Eq. (68), we see that the right-hand sides of Eqs. (63) and (64) are the desuspension of the right-hand sides of Eqs. (65) and (66), respectively. Hence,  $F$  is a  $L_\infty$ -morphism.

It is easy to see to see that the converse follows by reversing the above arguments.  $\square$

**A.6.  $L_\infty$ -morphisms and Central  $n$ -extensions.** Let  $(\mathfrak{g}, [\cdot, \cdot])$  be a Lie algebra and  $c: \Lambda^{n+1}\mathfrak{g} \rightarrow \mathbb{R}$  a degree  $n+1$  cocycle in the Chevalley-Eilenberg complex associated to  $\mathfrak{g}$ . A theorem of Baez and Crans [3, Thm. 55]: implies that this data gives a Lie  $n$ -algebra  $\widehat{\mathfrak{g}}_c$  whose underlying vector space is

$$\begin{aligned} L_0 &= \mathfrak{g}, \\ L_i &= 0 \quad 2-n \leq i \leq -1, \\ L_{1-n} &= \mathbb{R}, \end{aligned}$$

and whose only non-trivial multibrackets are

$$\begin{aligned} l_2(x_1, x_2) &= \begin{cases} [x_1, x_2] & \text{if } x_1, x_2 \in \mathfrak{g} \\ 0 & \text{otherwise} \end{cases} \\ l_{n+1}(x_1, \dots, x_{n+1}) &= \begin{cases} c(x_1, \dots, x_{n+1}) & \text{if } x_1, \dots, x_{n+1} \in \mathfrak{g} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

We call such Lie  $n$ -algebras **central  $n$ -extensions** of  $\mathfrak{g}$ .

**Proposition A.8.** *Let  $\mathfrak{g}$  be a Lie algebra,  $c \in \text{Hom}(\Lambda^{n+1}\mathfrak{g}, \mathbb{R})$  a  $(n+1)$ -cocycle, and  $\widehat{\mathfrak{g}}_c$  the corresponding central  $n$ -extension. If  $(L, l_k)$  is a Lie  $n$ -algebra satisfying property (P), then a collection of  $n$  skew-symmetric maps*

$$\begin{aligned} f_1 &: \mathfrak{g} \oplus \mathbb{R}[n-1] \rightarrow L \\ f_m &: \mathfrak{g}^{\otimes m} \rightarrow L, \quad |f_m| = 1 - m, \quad 2 \leq m \leq n \end{aligned}$$

*determine an  $L_\infty$ -morphism  $\bar{S}(\mathfrak{s}^{-1}\widehat{\mathfrak{g}}_c) \rightarrow \bar{S}(\mathfrak{s}^{-1}L)$  if and only if*

$$(69) \quad l_1 f_1(r) = 0 \quad \forall r \in \mathbb{R},$$

*and  $\forall x_i \in \mathfrak{g}$*

$$(70) \quad \sum_{1 \leq i < j \leq m} (-1)^{i+j+1} f_{m-1}([x_i, x_j], x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_m) \\ = l_1 f_m(x_1, \dots, x_m) + l_m(f_1(x_1), \dots, f_1(x_m)).$$

*for  $2 \leq m \leq n$  and*

$$(71) \quad \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+1} f_n([x_i, x_j], x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_{n+1}) + f_1 c(x_1, \dots, x_{n+1}) \\ = l_{n+1}(f_1(x_1), \dots, f_1(x_{n+1})).$$

*Proof.* Observe the similarity between the above formulas and those given in Cor. A.7 for a  $L_\infty$ -morphism from  $\mathfrak{g}$  to  $(L, l_k)$ . The only differences involve the map  $f_1$  and  $f_n$ . Let  $D$  denote the codifferential on  $\bar{S}(\mathfrak{s}^{-1}\widehat{\mathfrak{g}}_c)$ . We proceed as we did in the proof of Prop. A.6 and conclude that a coalgebra morphism  $F: \bar{S}(\mathfrak{s}^{-1}\widehat{\mathfrak{g}}_c) \rightarrow \bar{S}(\mathfrak{s}^{-1}L)$  is an  $L_\infty$ -morphism iff

$$\begin{aligned} Q_1 F_1(\mathfrak{s}^{-1}r) &= 0 \quad \forall r \in \mathbb{R}[n-1], \\ F_{m-1}^1 D_m^{m-1} &= Q_1^1 F_m^1 + Q_m^1 F_m^m \quad 2 \leq m \leq n, \end{aligned}$$

and

$$F_n^1 D_{n+1}^n + F_1^1 D_{n+1}^1 = Q_{n+1}^1 F_{n+1}^{n+1}.$$

Rewriting these in terms of structure maps  $(f_k)$  (cf. the proof of Cor. A.7), we obtain Eqs. (69), (70), and (71).  $\square$

Note that a central  $n$ -extension itself satisfies Property (P), so we have the following corollary:

**Corollary A.9.** *If  $[c] = [c'] \in H_{\text{CE}}^{n+1}(\mathfrak{g}, \mathbb{R})$ , then the central  $n$ -extensions  $\widehat{\mathfrak{g}}_c$  and  $\widehat{\mathfrak{g}}_{c'}$  are quasi-isomorphic.*

*Proof.* Let  $b: \Lambda^n \mathfrak{g} \rightarrow \mathbb{R}$  such that  $c' = c + \delta_{\text{CE}} b$ . Consider the collection of skew-symmetric maps:  $f_1 = \text{id}_{\mathfrak{g} \oplus \mathbb{R}[n-1]}$ ,  $f_k = 0$  for  $2 \leq k \leq n-1$  and  $f_n = b$ . Using Prop. A.8, it's easy to see these give an  $L_\infty$ -morphism  $\widehat{\mathfrak{g}}_c \rightarrow \widehat{\mathfrak{g}}_{c'}$ . Since  $f_1$  is the identity, it is clearly a quasi-isomorphism.  $\square$

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